REASONABLE ULTRAFILTERS, AGAIN

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Abstract. We continue investigations of *reasonable ultrafilters* on uncountable cardinals defined in Shelah [8]. We introduce stronger properties of ultrafilters and we show that those properties may be handled in λ–support iterations of reasonably bounding forcing notions. We use this to show that consistently there are reasonable ultrafilters on an inaccessible cardinal λ with generating system of size less than 2^λ. We also show how reasonable ultrafilters can be killed by forcing notions which have enough reasonable completeness to be iterated with λ–supports (and we show the appropriate preservation theorem).

0. Introduction

Reasonable ultrafilters were introduced in Shelah [8] in order to suggest a line of research that would repeat in some sense the beautiful theory created around the notion of *P–points on ω*. Most of the generalizations of P–points to uncountable cardinals in the literature goes into the direction of normal ultrafilters and large cardinals (see, e.g., Gitik [3]), but one may be interested in the opposite direction. If one wants to keep away from *normal ultrafilters on λ*, one may declare interest in ultrafilters which do not include all clubs and even demand that quotients by a closed unbounded subset of λ do not extend the club filter of λ. Such ultrafilters are called *weakly reasonable ultrafilters*, see 1.1, 1.2. But if we are interested in generalizing P–points, we have to consider also properties that would correspond to *any countable family of members of the ultrafilter has a pseudo-intersection in the ultrafilter*. The choice of the right property in the declared context of *very non-normal ultrafilters* is not clear, and the goal of the present paper is to show that the *very reasonable ultrafilters* suggested in Shelah [8] (see Definition 1.3 here) are very reasonable indeed, that is we may prove interesting theorems on them.

In the first section we recall some of the concepts and results presented in Shelah [8] and we introduce strong properties of generating systems (super and strong reasonability, see Definitions 1.11, 1.12) and we show that there may exist super reasonable systems generating ultrafilters (Propositions 1.15, 1.16).

In the next section we remind from [6] some properties of forcing notions relevant for λ–support iterations. We also improve in some sense a result of [6] and we show a preservation theorem for *nice double* $\mathbf{a}$*–bounding property* (Theorem 2.12).

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Then in the third section we show that super reasonable families generating ultrafilters will be at least strongly reasonable and will continue to generate ultrafilters after forcing with λ–support iterations of **A**–bounding forcing notions. Therefore, for an inaccessible cardinal λ, it is consistent that $2^\lambda = \lambda^{++}$ and there is a very reasonable ultrafilter generated by a family of size λ^+ (Corollary 3.4).

The fourth section shows that some technical inconveniences of the proofs from the third sections reflect the delicate nature of our concepts, not necessarily our lack of knowledge. We give an example of a nicely double **a**–bounding forcing notion which kills ultrafilters generated by systems from the ground model. Then we show that for an inaccessible cardinal λ, it is consistent that $2^\lambda = \lambda^{++}$ and there is no ultrafilter generated by a system of size λ^+ (see Corollary 3.4).

Notation: Our notation is rather standard and compatible with that of classical textbooks (like Jech [5]). In forcing we keep the older convention that *a stronger condition is the larger one.*

(1) Ordinal numbers will be denoted be the lower case initial letters of the Greek alphabet $(\alpha, \beta, \gamma, \delta \ldots)$ and also by i, j (with possible sub- and superscripts). Cardinal numbers will be called κ, λ, μ (with possible sub- and superscripts). λ **is always assumed to be regular, sometimes even strongly inaccessible**.

By χ we will denote a *sufficiently large* regular cardinal; $\mathcal{H}(\chi)$ is the family of all sets hereditarily of size less than χ. Moreover, we fix a well ordering $<^*_\chi$ of $\mathcal{H}(\chi)$.

(2) For two sequences η, ν we write $\nu \vartriangleleft \eta$ whenever ν is a proper initial segment of η, and $\nu \trianglelefteq \eta$ when either $\nu \vartriangleleft \eta$ or $\nu = \eta$. The length of a sequence η is denoted by $\mathrm{lh}(\eta)$.

(3) We will consider several games of two players. One player will be called *Generic* or *Complete* or just *COM*, and we will refer to this player as "she". Her opponent will be called *Antigeneric* or *Incomplete* or just *INC* and will be referred to as "he".

(4) For a forcing notion $\mathbb{P}$, $\Gamma_{\mathbb{P}}$ stands for the canonical $\mathbb{P}$–name for the generic filter in $\mathbb{P}$. With this one exception, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below (e.g., $\underset{\sim}{\tau}$, $\underset{\sim}{X}$). The weakest element of $\mathbb{P}$ will be denoted by $\emptyset_{\mathbb{P}}$ (and we will always assume that there is one, and that there is no other condition equivalent to it). We will also assume that all forcing notions under considerations are atomless.

By "λ–support iterations" we mean iterations in which domains of conditions are of size $\leq \lambda$. However, we will pretend that conditions in a λ–support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\zeta : \zeta < \zeta^* \rangle$ are total functions on ζ^* and for $p \in \lim(\bar{\mathbb{Q}})$ and $\alpha \in \zeta^* \setminus \mathrm{Dom}(p)$ we will let $p(\alpha) = \emptyset_{\underset{\sim}{\mathbb{Q}}_\alpha}$.

(5) For a filter D on λ, the family of all D–positive subsets of λ is called D^+. (So $A \in D^+$ if and only if $A \subseteq \lambda$ and $A \cap B \neq \emptyset$ for all $B \in D$.)

The club filter of λ is denoted by $\mathcal{D}_\lambda$.

1. More reasonable ultrafilters on λ

Here we recall some basic definitions and results from [8], and then we introduce even stronger properties of ultrafilters and/or generating systems. We also show that assumptions like $\diamondsuit_{S^{\lambda^+}_\lambda}$ imply the existence of such objects.

Definition 1.1 ([8, Def. 1.4])**.** We say that a uniform ultrafilter D on λ is *weakly reasonable* if for every non-decreasing unbounded function $f \in {}^{\lambda}\lambda$ there is a club C of λ such that

$$\bigcup\{[\delta, \delta + f(\delta)) : \delta \in C\} \notin D.$$

Observation 1.2 ([8, Obs. 1.5])**.** *Let D be a uniform ultrafilter on λ. Then the following conditions are equivalent:*

(A) *D is weakly reasonable,*

(B) *for every increasing continuous sequence $\langle \delta_\xi : \xi < \lambda \rangle \subseteq \lambda$ there is a club C^* of λ such that*

$$\bigcup \big\{ [\delta_\xi, \delta_{\xi+1}) : \xi \in C^* \big\} \notin D,$$

(C) *for every club C of λ the quotient D/C does not extend the filter generated by clubs of λ.*

Definition 1.3 ([8, Def. 2.5])**.** (1) Let $\mathbb{Q}^0_\lambda$ consist of all tuples $p = (C^p, \langle Z^p_\delta : \delta \in C^p \rangle, \langle d^p_\delta : \delta \in C^p \rangle)$ such that

(i) C^p is a club of λ consisting of limit ordinals only, and for $\delta \in C^p$:

(ii) $Z^p_\delta = \big[\delta, \min\big(C^p \setminus (\delta + 1)\big)\big)$ and

(iii) $d^p_\delta \subseteq \mathcal{P}(Z^p_\delta)$ is a proper ultrafilter on Z^p_δ.

(2) For $q \in \mathbb{Q}^0_\lambda$ we let

$$\mathrm{fil}(q) \stackrel{\mathrm{def}}{=} \big\{ A \subseteq \lambda : (\exists \varepsilon < \lambda)(\forall \delta \in C^q \setminus \varepsilon)(A \cap Z^q_\delta \in d^q_\delta) \big\},$$

and for a set $G^* \subseteq \mathbb{Q}^0_\lambda$ we let $\mathrm{fil}(G^*) \stackrel{\mathrm{def}}{=} \bigcup\{\mathrm{fil}(p) : p \in G^*\}$. We also define a binary relation $\leq^0$ on $\mathbb{Q}^0_\lambda$ by

$$p \leq^0 q \quad \text{if and only if} \quad \mathrm{fil}(p) \subseteq \mathrm{fil}(q).$$

(3) We say that an ultrafilter D on λ is *reasonable* if it is weakly reasonable (see 1.1) and there is a directed (with respect to $\leq^0$) set $G^* \subseteq \mathbb{Q}^0_\lambda$ such that $D = \mathrm{fil}(G^*)$. The family G^* may be called *the generating system for D*.

(4) An ultrafilter D on λ is said to be *very reasonable* if it is weakly reasonable and there is a $({<}\lambda^+)$–directed (with respect to $\leq^0$) set $G^* \subseteq \mathbb{Q}^0_\lambda$ such that $D = \mathrm{fil}(G^*)$.

Definition 1.4. Suppose that

(a) X is a non-empty set and e is an ultrafilter on X,

(b) d_x is an ultrafilter on a set Z_x (for $x \in X$).

We let

$$\bigoplus_{x \in X}^{e} d_x = \big\{ A \subseteq \bigcup_{x \in X} Z_x : \{x \in X : Z_x \cap A \in d_x\} \in e \big\}.$$

(Clearly, $\bigoplus\limits_{x \in X}^{e} d_x$ is an ultrafilter on $\bigcup\limits_{x \in X} Z_x$.)

Proposition 1.5 ([8, Prop. 2.9])**.** *Let $p, q \in \mathbb{Q}^0_\lambda$. Then the following are equivalent:*

(a) *$p \leq^0 q$,*

(b) *there is $\varepsilon < \lambda$ such that*

$$\big(\forall \alpha \in C^q \setminus \varepsilon\big)\big(\forall A \in d^q_\alpha\big)\big(\exists \beta \in C^p\big)\big(A \cap Z^p_\beta \in d^p_\beta\big),$$

(c) *there is* $\varepsilon < \lambda$ *such that*
if $\alpha \in C^q \setminus \varepsilon$, $\beta_0 = \sup\big(C^p \cap (\alpha+1)\big)$, $\beta_1 = \min\big(C^p \setminus \min(C^q \setminus (\alpha+1))\big)$, *then there is an ultrafilter* e *on* $[\beta_0, \beta_1) \cap C^p$ *such that*

$$d^q_\alpha = \big\{A \cap Z^q_\alpha : A \in \bigoplus^{e} \{d^p_\beta : \beta \in [\beta_0, \beta_1) \cap C^p\}\big\}.$$

Observation 1.6 (Compare [8, Prop. 2.3(4)])**.** *If* $p \in \mathbb{Q}^0_\lambda$, $A \subseteq \lambda$, *then there is* $q \in \mathbb{Q}^0_\lambda$ *such that* $p \leq^0 q$ *and either* $A \in \mathrm{fil}(q)$ *or* $\lambda \setminus A \in \mathrm{fil}(q)$.

Definition 1.7 ([8, Def. 2.10])**.** Let $p \in \mathbb{Q}^0_\lambda$. Suppose that $X \in [C^p]^\lambda$ and $C \subseteq C^p$ is a club of λ such that

if $\alpha < \beta$ are successive elements of C,
then $|[\alpha, \beta) \cap X| = 1$.

(In this situation we say that p *is restrictable to* $\langle X, C\rangle$.) We define *the restriction of* p *to* $\langle X, C\rangle$ as an element $q = p{\restriction}\langle X, C\rangle \in \mathbb{Q}^0_\lambda$ such that $C^q = C$, and if $\alpha < \beta$ are successive elements of C, $x \in [\alpha, \beta) \cap X$, then $Z^q_\alpha = [\alpha, \beta)$ and $d^q_\alpha = \{A \subseteq Z^q_\alpha : A \cap Z^p_x \in d^p_x\}$.

Proposition 1.8 ([8, Prop. 2.11])**.** (1) *Assume that* $G^* \subseteq \mathbb{Q}^0_\lambda$ *is* $\leq^0$*–directed and* $\leq^0$*–downward closed,* $p \in G^*$, $X \in [C^p]^\lambda$ *and* $C \subseteq C^p$ *is a club of* λ *such that* p *is restrictable to* $\langle X, C\rangle$. *If* $\bigcup\limits_{x \in X} Z^p_x \in \mathrm{fil}(G^*)$, *then* $p{\restriction}\langle X, C\rangle \in G^*$.

(2) *If* $G^* \subseteq \mathbb{Q}^0_\lambda$ *is* $\leq^0$*–directed and* $|G^*| \leq \lambda$, *then* G^* *has a* $\leq^0$*–upper bound. (Hence, in particular,* $\mathrm{fil}(G^*)$ *is not an ultrafilter.)*

The various definitions of super reasonable ultrafilters introduced below are motivated by the proof of "the Sacks forcing preserves P–points". In that proof, a fusion sequence is constructed so that at a stage $n < \omega$ of the construction one deals with *finitely many* nodes in a condition (the nodes that are declared to be kept). We would like to carry out this kind of argument, e.g., for forcing notions used in [7, B.8.3, B.8.5], but now we got to deal with $< \lambda$ nodes in a tree, and the ultrafilter we try to preserve is not that complete. So what do we do? We deal with *finitely* many nodes at a time eventually taking care of everybody. One can think that in the definition below the set I_α is the set of nodes we have to keep and the finite sets $u_{\alpha,i}$ are the nodes taken care of at a substage i.

Definition 1.9. (1) Let $\mathbb{Q}^*_\lambda$ be the family of all sets r such that

(a) members or r are triples (α, Z, d) such that $\alpha < \lambda$, $Z \subseteq [\alpha, \lambda)$, $|Z| < \lambda$ and d is an ultrafilter on Z, and
(b) $\big(\forall \xi < \lambda\big)\big(|\{(\alpha, Z, d) \in r : \alpha = \xi\}| < \lambda\big)$, and $|r| = \lambda$.

For for $r \in \mathbb{Q}^*_\lambda$ we define

$$\mathrm{fil}^*(r) = \big\{A \subseteq \lambda : \big(\exists \varepsilon < \lambda\big)\big(\forall (\alpha, Z, d) \in r\big)\big(\varepsilon \leq \alpha \ \Rightarrow\ A \cap Z \in d\big)\big\},$$

and we define a binary relation $\leq^*$ on $\mathbb{Q}^*_\lambda$ by
$r_1 \leq^* r_2$ if and only if ($r_1, r_2 \in \mathbb{Q}^*_\lambda$ and) $\mathrm{fil}^*(r_1) \subseteq \mathrm{fil}^*(r_2)$.

(2) For a directed (with respect to $\leq^*$) set $G_* \subseteq \mathbb{Q}^*_\lambda$ we let $\mathrm{fil}^*(G_*) = \bigcup \big\{\mathrm{fil}^*(r) : r \in G_*\big\}$.

(3) We say that an $r \in \mathbb{Q}^*_\lambda$ is *strongly disjoint* if and only if

- $\big(\forall \xi < \lambda\big)\big(|\{(\alpha, Z, d) \in r : \alpha = \xi\}| < 2\big)$, and
- $\big(\forall (\alpha_1, Z_1, d_1), (\alpha_2, Z_2, d_2) \in r\big)\big(\alpha_1 < \alpha_2 \ \Rightarrow\ Z_1 \subseteq \alpha_2\big)$.

(4) For $p \in \mathbb{Q}^0_\lambda$ we let $\#(p) = \{(\alpha, Z^p_\alpha, d^p_\alpha) : \alpha \in C^p\}$.

Observation 1.10. (1) *If* $p \in \mathbb{Q}^0_\lambda$ *then* $\#(p) \in \mathbb{Q}^*_\lambda$ *is strongly disjoint and* $\mathrm{fil}(p) = \mathrm{fil}^*(\#(p))$.

(2) *Let* $r, s \in \mathbb{Q}^*_\lambda$. *Then* $r \leq^* s$ *if and only if there is* $\varepsilon < \lambda$ *such that*

$$\big(\forall(\alpha, Z, d) \in s\big)\big(\forall A \in d\big)\big(\alpha > \varepsilon \ \Rightarrow\ \big(\exists(\alpha', Z', d') \in r\big)\big(A \cap Z' \in d'\big)\big).$$

Definition 1.11. Let $G^* \subseteq \mathbb{Q}^0_\lambda$ and let $\bar\mu = \langle \mu_\alpha : \alpha < \lambda\rangle$ be a sequence of cardinals, $2 \leq \mu_\alpha \leq \lambda$ for $\alpha < \lambda$.

(1) We define a game $\Game^{\boxplus}_{\bar\mu}(G^*)$ between two players, COM and INC. A play of $\Game^{\boxplus}_{\bar\mu}(G^*)$ lasts λ steps and at a stage $\alpha < \lambda$ of the play the players choose $I_\alpha, i_\alpha, \bar u_\alpha$ and $\langle r_{\alpha,i}, r'_{\alpha,i}, (\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : i < i_\alpha\rangle$ applying the following procedure.

- First, INC chooses a non-empty set I_α of cardinality $< \mu_\alpha$ and an enumeration $\bar u_\alpha = \langle u_{\alpha,i} : i < i_\alpha\rangle$ of $[I_\alpha]^{<\omega}$ (so $i_\alpha < \mu_\alpha \cdot \aleph_0$).
- Next the two players play a subgame of length i_α. In the $i^{\rm th}$ move of the subgame,
  - (a) COM chooses $r_{\alpha,i} \in G^*$, and then
  - (b) INC chooses $r'_{\alpha,i} \in G^*$ such that $r_{\alpha,i} \leq^0 r'_{\alpha,i}$, and finally
  - (c) COM picks $(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) \in \#\big(r'_{\alpha,i}\big)$.

In the end of the play COM wins if and only if

($\boxplus$) there is $r \in G^*$ such that for every $\bar j \in \prod\limits_{\alpha<\lambda} I_\alpha$ we have

$$\{(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : \alpha < \lambda,\ j_\alpha \in u_{\alpha,i} \text{ and } i < i_\alpha\} \leq^* \#(r).$$

A game $\Game^{\boxminus}_{\bar\mu}(G^*)$ is defined similarly to $\Game^{\boxplus}_{\bar\mu}(G^*)$ except that ($\boxplus$) is weakened to

($\boxminus$) for every $\bar j \in \prod\limits_{\alpha<\lambda} I_\alpha$ the set $\bigcup\{Z_{\alpha,i} : \alpha < \lambda,\ i < i_\alpha \text{ and } j_\alpha \in u_{\alpha,i}\}$ belongs to $\mathrm{fil}(G^*)$.

(2) We say that the family G^* is *$\bar\mu$–super reasonable* (*$\bar\mu$–super$^-$ reasonable*, respectively) if
  - (i) G^* is $({<}\lambda^+)$–directed (with respect to $\leq^0$), and
  - (ii) if $s \in G^*$, $r \in \mathbb{Q}^0_\lambda$ and for some $\alpha < \lambda$ we have $C^r \setminus \alpha = C^s \setminus \alpha$ and $d^r_\beta = d^s_\beta$ for $\beta \in C^r \setminus \alpha$, then $r \in G^*$, and
  - (iii) INC has no winning strategy in the game $\Game^{\boxplus}_{\bar\mu}(G^*)$ ($\Game^{\boxminus}_{\bar\mu}(G^*)$, respectively).

(3) We say that a uniform ultrafilter D on λ is *$\bar\mu$–super reasonable* (*$\bar\mu$–super$^-$ reasonable*, respectively) if there is a $\bar\mu$–super reasonable ($\bar\mu$–super$^-$ reasonable, respectively) set $G^* \subseteq \mathbb{Q}^0_\lambda$ such that $D = \mathrm{fil}(G^*)$.

(4) If $\mu_\alpha = \lambda$ for all $\alpha < \lambda$, then we omit $\bar\mu$ and say just *super reasonable* or *super$^-$ reasonable* (in reference to both ultrafilters on λ and families $G^* \subseteq \mathbb{Q}^0_\lambda$). Also in this case we may write $\Game^{\boxplus}$ instead of $\Game^{\boxplus}_{\bar\mu}$.

Definition 1.12. Let $G^* \subseteq \mathbb{Q}^0_\lambda$ be directed with respect to $\leq^0$ and let $\bar\mu = \langle \mu_\alpha : \alpha < \lambda\rangle$ be a sequence of cardinals, $2 \leq \mu_\alpha \leq \lambda$ for $\alpha < \lambda$.

(1) A game $\Game^{\oplus}_{\bar\mu}(G^*)$ between two players, COM and INC is defined as follows. A play of $\Game^{\oplus}_{\bar\mu}(G^*)$ lasts λ steps and at a stage $\alpha < \lambda$ of the play the players choose $I_\alpha, i_\alpha, \bar u_\alpha$ and $\langle r_{\alpha,i}, \delta_{\alpha,i}, (\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : i < i_\alpha\rangle$ applying the following procedure.

- First, INC chooses a non-empty set I_α of cardinality $< \mu_\alpha$, and then COM chooses $i_\alpha < \lambda$ and a sequence $\bar{u}_\alpha = \langle u_{\alpha,i} : i < i_\alpha \rangle$ of non-empty finite subsets of I_α such that $I_\alpha = \bigcup\limits_{i<i_\alpha} u_{\alpha,i}$.
- Next the two players play a subgame of length i_α. In the $i^{\rm th}$ move of the subgame,
  (a) COM chooses $r_{\alpha,i} \in G^*$ and then
  (b) INC chooses $\delta_{\alpha,i} < \lambda$, and finally
  (c) COM picks $(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) \in \#\big(r_{\alpha,i}\big)$ such that $\beta_{\alpha,i} > \delta_{\alpha,i}$.

In the end of the play COM wins if and only if

($\oplus$) there is $r \in G^*$ such that for every $\bar{j} \in \prod\limits_{\alpha<\lambda} I_\alpha$ we have

$$\{(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : \alpha < \lambda,\ j_\alpha \in u_{\alpha,i} \text{ and } i < i_\alpha\} \leq^* \#(r).$$

A game $\Game^{\ominus}_{\bar{\mu}}(G^*)$ is defined similarly to $\Game^{\oplus}_{\bar{\mu}}(G^*)$ except that ($\oplus$) is weakened to

($\ominus$) for every $\bar{j} \in \prod\limits_{\alpha<\lambda} I_\alpha$ the set $\bigcup\{Z_{\alpha,i} : \alpha < \lambda,\ i < i_\alpha \text{ and } j_\alpha \in u_{\alpha,i}\}$ belongs to $\mathrm{fil}(G^*)$.

(2) If $G^* \subseteq \mathbb{Q}^0_\lambda$ is $({<}\lambda^+)$–directed (with respect to $\leq^0$) and INC has no winning strategy in the game $\Game^{\oplus}_{\bar{\mu}}(G^*)$, then we say that G^* is *$\bar{\mu}$–strongly reasonable.* Also, G^* is said to be *$\bar{\mu}$–strongly$^-$ reasonable* if it is $({<}\lambda^+)$–directed and INC has no winning strategy in the game $\Game^{\ominus}_{\bar{\mu}}(G^*)$.

(3) We say that a uniform ultrafilter D on λ is *$\bar{\mu}$–strongly reasonable* (*$\bar{\mu}$–strongly$^-$ reasonable*, respectively) if there is a $\bar{\mu}$–strongly reasonable ($\bar{\mu}$–strongly$^-$ reasonable, respectively) set $G^* \subseteq \mathbb{Q}^0_\lambda$ such that $D = \mathrm{fil}(G^*)$. If $\mu_\alpha = \lambda$ for all $\alpha < \lambda$, then we omit $\bar{\mu}$ and say just *strongly reasonable* or *strongly$^-$ reasonable*.

Observation 1.13. *Assume that $2 \leq \mu_\alpha \leq \kappa_\alpha \leq \lambda$ for $\alpha < \lambda$ and $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$, $\bar{\kappa} = \langle \kappa_\alpha : \alpha < \lambda \rangle$. Then for a family $G^* \subseteq \mathbb{Q}^0_\lambda$ and/or a uniform ultrafilter D on λ the following implications hold.*

$$\begin{array}{ccccc}
\bar{\kappa}\text{–super reasonable} & \Rightarrow & \bar{\mu}\text{–super reasonable} & \Rightarrow & \bar{\mu}\text{–strongly reasonable} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\bar{\kappa}\text{–super}^- \text{ reasonable} & \Rightarrow & \bar{\mu}\text{–super}^- \text{ reasonable} & \Rightarrow & \bar{\mu}\text{–strongly}^- \text{ reasonable}
\end{array}$$

Proposition 1.14. *Assume that $2 \leq \mu_\alpha \leq \lambda$ for $\alpha < \lambda$ and $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$. If a uniform ultrafilter D on λ is $\bar{\mu}$–strongly$^-$ reasonable, then it is very reasonable.*

Proof. Let $G^* \subseteq \mathbb{Q}^0_\lambda$ be a $\bar{\mu}$–strongly$^-$ reasonably family such that $D = \mathrm{fil}(G^*)$.

Let $f \in {}^\lambda\lambda$ be a non-decreasing unbounded function. Consider the following strategy $\mathbf{st}(f)$ for INC in $\Game^{\ominus}_{\bar{\mu}}(G^*)$. The strategy $\mathbf{st}(f)$ instructs INC to construct aside an increasing continuous sequence $\langle \gamma_\alpha : \alpha < \lambda \rangle \subseteq \lambda$ and at a stage $\alpha < \lambda$ of the play, when

$$\big\langle I_\xi, i_\xi, \bar{u}_\xi, \langle r_{\xi,i}, \delta_{\xi,i}, (\beta_{\xi,i}, Z_{\xi,i}, d_{\xi,i}) : i < i_\xi \rangle : \xi < \alpha \big\rangle$$

is the result of the play so far, then

- if α is limit, then $\gamma_\alpha = \sup(\gamma_\xi : \xi < \alpha)$,
- if α is not limit, then $\gamma_\alpha = \sup\big(\bigcup\{Z_{\xi,i} : i < i_\xi,\ \xi < \alpha\}\big) + 1$.

Now (at the stage α) $\mathbf{st}(f)$ instructs INC to choose $I_\alpha = \{0\}$ and then (after COM picks $i_\alpha, \bar{u}_\alpha$) he is instructed to play in the subgame of this stage as follows. At

stage $i < i_\alpha$, after COM has picked $r_{\alpha,i}$, INC lets

$$\delta_{\alpha,i} = \gamma_\alpha + f(\gamma_\alpha) + \sup\big(\bigcup\{Z_{\alpha,j} : j < i\}\big) + 890.$$

(After this COM chooses $(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) \in \#\big(r_{\alpha,i}\big)$ with $\beta_{\alpha,i} > \delta_{\alpha,i}$.)

The strategy $\mathbf{st}(f)$ cannot be the winning one for INC, so there is a play

$$\big\langle I_\alpha, i_\alpha, \bar{u}_\alpha, \langle r_{\alpha,i}, \delta_{\alpha,i}, (\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : i < i_\alpha\rangle : \alpha < \lambda\big\rangle$$

of $\eth^{\ominus}_{\bar{\mu}}(G^*)$ in which INC follows $\mathbf{st}(f)$ but

$$A^* \stackrel{\rm def}{=} \bigcup\big\{Z_{\alpha,i} : \alpha < \lambda,\ i < i_\alpha\big\} \in \mathrm{fil}(G^*) = D$$

(note that necessarily $u_{\alpha,i} = I_\alpha = \{0\}$). It follows from the choice of $\gamma_\alpha, \delta_{\alpha,i}$ that for each $\alpha < \lambda$

$$[\gamma_\alpha, \gamma_\alpha + f(\gamma_\alpha)) \cap \bigcup\big\{Z_{\xi,i} : \xi < \lambda,\ i < i_\xi\big\} = \emptyset,$$

and hence also $\bigcup\big\{[\gamma_\alpha, \gamma_\alpha + f(\gamma_\alpha)) : \alpha < \lambda\big\} \cap A^* = \emptyset$. Consequently $\bigcup\big\{[\gamma_\alpha, \gamma_\alpha + f(\gamma_\alpha)) : \alpha < \lambda\big\} \notin D$ and one can easily finish the proof. □

Proposition 1.15. *Assume $\lambda = \lambda^{<\lambda}$ and $\diamondsuit_{S^{\lambda^+}_\lambda}$ holds. There exists a sequence $\langle r_\xi : \xi < \lambda^+\rangle \subseteq \mathbb{Q}^0_\lambda$ such that*

(i) $(\forall \xi < \zeta < \lambda^+)(r_\xi \leq^0 r_\zeta)$, *and*

(ii) *the family*

$$G^* \stackrel{\rm def}{=} \big\{r \in \mathbb{Q}^0_\lambda : (\exists \xi < \lambda^+)(r \leq^0 r_\xi)\big\}$$

is super reasonable and $\mathrm{fil}(G^*)$ *is an ultrafilter on λ.*

Proof. For $\alpha < \lambda$ let X^1_α be the set of all legal plays of $\eth^{\boxplus}(\mathbb{Q}^0_\lambda)$ of the form

$(\odot)^1_\alpha$ $\quad\big\langle I_\gamma, i_\gamma, \bar{u}_\gamma, \langle r_{\gamma,i}, r'_{\gamma,i}, (\beta_{\gamma,i}, Z_{\gamma,i}, d_{\gamma,i}) : i < i_\gamma\rangle : \gamma < \alpha\big\rangle$

where each I_γ (for $\gamma < \alpha$) is an ordinal below λ. Also let $X^1 = \bigcup\limits_{\alpha<\lambda} X^1_\alpha$. Next, for $\alpha < \lambda$, $I < \lambda$ and an enumeration $\bar{u} = \langle u_j : j < i\rangle$ of $[I]^{<\omega}$ let $X^2_{\alpha,I,\bar{u}}$ be the set of all legal plays of $\eth^{\boxplus}(\mathbb{Q}^0_\lambda)$ of the form

$(\odot)^2_{\alpha,I,\bar{u}}$ $\quad\bar{\sigma}^\frown\langle(I, i, \bar{u})\rangle^\frown\langle r_j, r'_j, (\beta_j, Z_j, d_j) : j < j^*\rangle^\frown\langle r\rangle$,

where $\bar{\sigma} \in X^1_\alpha$, $j^* < i$ (and $\langle r_j, r'_j, (\beta_j, Z_j, d_j) : j < j^*\rangle^\frown\langle r\rangle$ is a legal partial play of the subgame of level α; in particular $r_j, r'_j, r \in \mathbb{Q}^0_\lambda$). Also let

$$X^2 = \bigcup\big\{X^2_{\alpha,I,\bar{u}} : \alpha < \lambda\ \&\ 0 < I < \lambda\ \&\ \bar{u} = \langle u_j : j < i\rangle \text{ is an enumeration of } [I]^{<\omega}\big\}.$$

Any strategy for INC in $\eth^{\boxplus}(\mathbb{Q}^0_\lambda)$ can be interpreted as a function $\mathbf{st}$ such that

$(\odot)^3$ the domain of $\mathbf{st}$ is $X^1 \cup X^2$,

$(\odot)^4$ if $\bar{\sigma} \in X^1_\alpha$, $\alpha < \lambda$, then $\mathbf{st}(\bar{\sigma}) = (I, i, \bar{u})$ for some $I < \lambda$ and an enumeration $\bar{u} = \langle u_j : j < i\rangle$ of $[I]^{<\omega}$,

$(\odot)^5$ if $\bar{\sigma} \in X^2_{\alpha,I,\bar{u}}$, $\alpha < \lambda$, $I < \lambda$, $\bar{u} = \langle u_j : j < i\rangle = [I]^{<\omega}$, and $\bar{\sigma} = \bar{\sigma}_0{}^\frown\langle(I, i, \bar{u})^\frown\langle r_j, r'_j, (\beta_j, Z_j, d_j) : j < j^*\rangle^\frown\langle r\rangle$, then $\mathbf{st}(\bar{\sigma}) \in \mathbb{Q}^0_\lambda$ is such that $r \leq^0 \mathbf{st}(\bar{\sigma})$.

Below, whenever we say *a strategy for INC* we mean a function $\mathbf{st}$ satisfying $(\odot)^3$–$(\odot)^5$.

Since $|\mathbb{Q}^0_\lambda| = 2^\lambda = \lambda^+$, we may pick a bijection $\pi_0 : \mathbb{Q}^0_\lambda \xrightarrow{1-1} \lambda^+$ and for $\xi < \lambda^+$ let $\mathcal{X}_\xi$ consist of all $\bar{\sigma} \in \mathcal{X}^1 \cup \mathcal{X}^2$ such that $\pi_0(r) < \xi$ for all elements $r \in \mathbb{Q}^0_\lambda$ involved

in the representation of $\bar{\sigma}$ as in $(\odot)^1, (\odot)^2$. We also let $\mathcal{Y}_\xi$ consist of all pairs $(\bar{\sigma}, a)$ such that

- $\bar{\sigma} \in \mathcal{X}_\xi$ and $a = \mathbf{st}(\bar{\sigma})$ for some strategy $\mathbf{st}$ of INC, and
- if $\bar{\sigma} \in X^2$ (and so $a \in \mathbb{Q}^0_\lambda$) then $\pi_0(a) < \xi$.

Note that $|\mathcal{X}_\xi| \leq \lambda$ and $|\mathcal{Y}_\xi| \leq \lambda$ (for each $\xi < \lambda^+$). Put $\mathcal{Y} = \bigcup\limits_{\xi<\lambda^+} \mathcal{Y}_\xi$. Plainly $|\mathcal{Y}| = \lambda^+$ so we may fix a bijection $\pi_1 : \lambda^+ \xrightarrow{\text{onto}} \mathcal{Y}$. Let $C = \{\xi < \lambda^+ : \pi_1[\xi] = \mathcal{Y}_\xi\}$; it is a club of λ^+.

Let $\langle A_\zeta : \zeta < \lambda^+\rangle$ list all subsets of λ and let $\langle B_\zeta : \zeta \in S^{\lambda^+}_\lambda\rangle$ be a diamond sequence on $S^{\lambda^+}_\lambda$. By induction on $\xi < \lambda^+$ we choose a $\leq^0$–increasing sequence $\langle r_\xi : \xi < \lambda^+\rangle \subseteq \mathbb{Q}^0_\lambda$ applying the following procedure. Assume $\xi < \lambda^+$ and we have constructed $\langle r_\zeta : \zeta < \xi\rangle$.

Case 0: $\xi = 0$.
We let r_0 be the $<^*_\chi$–first member of $\mathbb{Q}^0_\lambda$.

Case 1: $\xi = \zeta + 1$.
Pick $r_\xi \in \mathbb{Q}^0_\lambda$ such that $r_\zeta \leq^0 r_\xi$ and either $A_\zeta \in \mathrm{fil}(r_\xi)$ or $\lambda \setminus A_\zeta \in \mathrm{fil}(r_\xi)$ (remember Observation 1.6).

Case 2: ξ is a limit ordinal, $\mathrm{cf}(\xi) < \lambda$.
Pick $r_\xi \in \mathbb{Q}^0_\lambda$ such that $(\forall \zeta < \xi)(r_\zeta \leq^0 r_\xi)$ (exists by Proposition 1.8(2)).

Case 3: ξ is a limit ordinal, $\mathrm{cf}(\xi) = \lambda$.
Now we ask if

$(\odot)^6_\xi$ $\xi \in C$ and $(\forall \zeta < \xi)(\pi_0(r_\zeta) < \xi)$ and there is a strategy $\mathbf{st}$ for INC in $\Game^{\boxplus}(\mathbb{Q}^0_\lambda)$ such that $\pi_1[B_\xi] = \mathbf{st} \cap \mathcal{Y}_\xi = \mathbf{st}{\restriction}\mathcal{X}_\xi$.

If the answer to $(\odot)^6_\xi$ is negative, then we choose $r_\xi \in \mathbb{Q}^0_\lambda$ as in Case 2.

Suppose now that the answer to $(\odot)^6_\xi$ is positive (so in particular $\xi \in C$) and $\mathbf{st}$ is a strategy for INC such that $\pi_1[B_\xi] = \mathbf{st} \cap \mathcal{Y}_\xi = \mathbf{st}{\restriction}\mathcal{X}_\xi$. Let $\bar{\xi} = \langle \xi_\alpha : \alpha < \lambda\rangle$ be an increasing continuous sequence cofinal in ξ. Consider a play

$$\bar{\sigma} = \big\langle I_\alpha, i_\alpha, \bar{u}_\alpha, \langle r_{\alpha,i}, r'_{\alpha,i}, (\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : i < i_\alpha\rangle : \alpha < \lambda\big\rangle$$

of $\Game^{\boxplus}(\mathbb{Q}^0_\lambda)$ in which INC follows the strategy $\mathbf{st}$ and COM proceeds as follows. When playing $\Game^{\boxplus}(\mathbb{Q}^0_\lambda)$, at step $i < i_\alpha$ of the subgame of level $\alpha < \lambda$ (of $\Game^{\boxplus}(\mathbb{Q}^0_\lambda)$) COM chooses $r_{\alpha,i} = r_{\xi_\alpha}$ and then, after INC determines $r'_{\alpha,i}$ by $\mathbf{st}$, she picks the $<^*_\chi$–first $(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) \in \#(r'_{\alpha,i})$ satisfying:

$(\odot)^7_{\xi,\alpha,i}$ $(\forall \gamma \leq \alpha)(\forall A \in d_{\alpha,i})(\exists \delta \in C^{r_{\xi_\gamma}})(A \cap Z^{r_{\xi_\gamma}}_\delta \in d^{r_{\xi_\gamma}}_\delta)$ (remember 1.5) and

$(\odot)^8_{\xi,\alpha,i}$ $(\forall \gamma < \alpha)(\forall j < i_\gamma)(Z_{\gamma,j} \subseteq \beta_{\alpha,i})$ and $(\forall j < i)(Z_{\alpha,i} \subseteq \beta_{\alpha,i})$.

The above rules fully determine the play $\bar{\sigma}$ and it should be clear that $\bar{\sigma}{\restriction}\alpha \in \mathcal{X}_\xi$ for each $\alpha < \lambda$. Note that $\bar{\sigma}$ depends on B_ξ and $\bar{\xi}$ only (and not on $\mathbf{st}$, provided it is as required by $(\odot)^6_\xi$).

By the demands $(\odot)^8_{\xi,\alpha,i}$, we may choose an increasing continuous sequence $\langle \gamma_\alpha : \alpha < \lambda\rangle \subseteq \lambda$ such that $\gamma_0 = 0$ and $(\forall \alpha < \lambda)(\forall i < i_\alpha)(Z_{\alpha,i} \subseteq [\gamma_\alpha, \gamma_{\alpha+1}))$. Now, for $\alpha < \lambda$ choose an ultrafilter e_α on i_α such that

$(\odot)^9_{\xi,\alpha}$ $\big(\forall j \in I_\alpha\big)\big(\{i < i_\alpha : j \in u_{\alpha,i}\} \in e_\alpha\big)$

and let d_α be an ultrafilter on $[\gamma_\alpha, \gamma_{\alpha+1})$ such that

$(\odot)^{10}_{\xi,\alpha}$ $\bigoplus\limits^{e_\alpha} \big\{d_{\alpha,i} : i < i_\alpha\big\} \subseteq d_\alpha$.

Now let $r_\xi \in \mathbb{Q}^0_\lambda$ be such that

- $C^{r_\xi} = \{\gamma_\alpha : \alpha < \lambda\}$, and
- if $\delta = \gamma_\alpha$, then $Z^{r_\xi}_\delta = [\gamma_\alpha, \gamma_{\alpha+1})$ and $d^{r_\xi}_\delta = d_\alpha$.

One easily verifies that $r_{\xi_\alpha} \leq^0 r_\xi$ for all $\alpha < \lambda$ (remember $(\odot)^7$ and the choice of d_α; use 1.5) and so $r_\zeta \leq^0 r_\xi$ for all $\zeta < \xi$. It follows from $(\odot)^9_{\xi,\alpha}$ and $(\odot)^{10}_{\xi,\alpha}$ that

$$(\odot)^{11}_\xi \quad \big(\forall \bar{j} \in \prod_{\alpha<\lambda} I_\alpha\big)\big(\{(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : \alpha < \lambda \;\&\; j_\alpha \in u_{\alpha,i} \;\&\; i < i_{\alpha,i}\} \leq^* \#(r_\xi)\big).$$

After the construction of $\langle r_\xi : \xi < \lambda^+\rangle$ is carried out we let

$$G^* = \{r \in \mathbb{Q}^0_\lambda : (\exists \xi < \lambda^+)(r \leq^0 r_\xi)\}.$$

Plainly, G^* satisfies demands (i) and (ii) of 1.11(2) and $\mathrm{fil}(G^*)$ is an ultrafilter on λ (remember Case 1 of the construction). We should argue that INC has no winning strategy in $\eth^{\boxplus}(G^*)$. To this end suppose that $\mathbf{st}^{\boxplus}$ is a strategy of INC in $\eth^{\boxplus}(G^*)$. Pick $\xi \in S^{\lambda^+}_\lambda \cap C$ such that $(\forall \zeta < \xi)(\pi_0(r_\zeta) < \xi)$ and $\pi_1[B_\xi] = \mathbf{st}^{\boxplus} \cap \mathcal{Y}_\xi = \mathbf{st}^{\boxplus}{\restriction}\mathcal{X}_\xi$. Then when choosing r_ξ we gave a positive answer to $(\odot)^6_\xi$ and we constructed a play $\bar\sigma$ of $\eth^{\boxplus}(\mathbb{Q}^0_\lambda)$. In that play, INC follows $\mathbf{st}^{\boxplus}$ and COM chooses members of G^*, so it is a play of $\eth^{\boxplus}(G^*)$. Now the condition $(\odot)^{11}_\xi$ means that r_ξ witnesses that COM wins the play $\bar\sigma$ and consequently $\mathbf{st}^{\boxplus}$ is not a winning strategy for INC. $\square$

Proposition 1.16. *Let* $\mathbb{Q}^0_\lambda = (\mathbb{Q}^0_\lambda, \leq^0)$.

(1) $\mathbb{Q}^0_\lambda$ *is a* $({<}\lambda^+)$*–complete forcing notion of size* $2^{2^{<\lambda}}$.
(2) $\Vdash_{\mathbb{Q}^0_\lambda}$ *“ $\eth_{\mathbb{Q}^0_\lambda}$ is a super reasonable family and* $\mathrm{fil}(\eth_{\mathbb{Q}^0_\lambda})$ *is an ultrafilter ”.*

Proof. (1) Should be clear; see also Proposition 1.8(2).
(2) By the completeness of $\mathbb{Q}^0_\lambda$, forcing with it does not add new subsets of λ, and

$$\Vdash_{\mathbb{Q}^0_\lambda} \text{“ } \mathrm{fil}(\Gamma_{\mathbb{Q}^0_\lambda}) \text{ is a uniform ultrafilter on } \lambda \text{ ”}.$$

It should also be clear that $\Gamma_{\mathbb{Q}^0_\lambda}$ satisfies the demand of Definition 1.11(2)(ii) (in $\mathbf{V}^{\mathbb{Q}^0_\lambda}$). Let us argue that

$$\Vdash_{\mathbb{Q}^0_\lambda} \text{“ INC has no winning strategy in } \eth^{\boxplus}(\Gamma_{\mathbb{Q}^0_\lambda}) \text{ ”}$$

and to this end suppose $p \in \mathbb{Q}^0_\lambda$ and $\underset{\sim}{\mathbf{st}}$ is a $\mathbb{Q}^0_\lambda$–name such that

$$p \Vdash_{\mathbb{Q}^0_\lambda} \text{“ } \underset{\sim}{\mathbf{st}} \text{ is a strategy of INC in } \eth^{\boxplus}(\Gamma_{\mathbb{Q}^0_\lambda}) \text{ ”}.$$

We are going to construct a condition $q \in \mathbb{Q}^0_\lambda$ stronger than p and a play $\bar\sigma$ of $\eth^{\boxplus}(\mathbb{Q}^0_\lambda)$ such that

$$q \Vdash_{\mathbb{Q}^0_\lambda} \text{“ } \bar\sigma \text{ is a play of } \eth^{\boxplus}(\Gamma_{\mathbb{Q}^0_\lambda}) \text{ in which INC follows } \underset{\sim}{\mathbf{st}} \text{ but COM wins”}.$$

Let X^1, X^2 be defined as in the proof of 1.15 (see $(\odot)^1_\alpha$, $(\odot)^2_{\alpha,I,\bar u}$ there). We may assume that

$$p \Vdash_{\mathbb{Q}^0_\lambda} \text{“ } \underset{\sim}{\mathbf{st}} \text{ is a function satisfying } (\odot)^3\text{–}(\odot)^5 \text{ of the proof of 1.15 ”}.$$

By induction on $\alpha < \lambda$ we choose conditions $p_\alpha \in \mathbb{Q}^0_\lambda$ and partial plays $\bar\sigma_\alpha \in X^1_\alpha$ so that

$(\boxdot)_1$ $p \leq^0 p_\alpha \leq^0 p_\beta$ and $\bar\sigma_\alpha \lhd \bar\sigma_\beta$ for $\alpha < \beta < \lambda$,
$(\boxdot)_2$ $p_\alpha \Vdash_{\mathbb{Q}^0_\lambda}$ “ $\bar\sigma_\alpha$ is a play of $\eth^{\boxplus}(\Gamma_{\mathbb{Q}^0_\lambda})$ in which INC uses $\underset{\sim}{\mathbf{st}}$ ”,

$(\boxdot)_3$ if $\bar{\sigma}_\alpha = \big\langle I_\gamma, i_\gamma, \bar{u}_\gamma, \langle r_{\gamma,i}, r'_{\gamma,i}, (\beta_{\gamma,i}, Z_{\gamma_i}, d_{\gamma,i}) : i < i_\gamma\rangle : \gamma < \alpha\big\rangle$, then for every $\gamma < \delta < \alpha$ and $j < i < i_\gamma$ we have

$$r'_{\gamma,i} \leq^0 p_\alpha \quad \text{and} \quad Z_{\gamma,j} \subseteq \beta_{\gamma,i} \quad \text{and} \quad Z_{\gamma,j} \subseteq \beta_{\delta,0}.$$

Suppose that $\alpha = \alpha^* + 1$ and we have determined $p_{\alpha^*}, \bar{\sigma}_{\alpha^*}$. Pick $p'_\alpha \geq^0 p_{\alpha^*}$ and $I_\alpha, i_\alpha, \bar{u}_\alpha$ such that $p'_\alpha \Vdash \underset{\sim}{\mathbf{st}}(\bar{\sigma}_{\alpha^*}) = (I_\alpha, i_\alpha, \bar{u}_\alpha)$. Now choose inductively $p^i_\alpha, r_{\alpha,i}, r'_{\alpha,i}$ and $(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i})$ for $i < i_\alpha$ so that for each $i < j < i_\alpha$ we have

$(\boxdot)_4$ (i) $p^0_\alpha = p'_\alpha$, $p^i_\alpha \leq^0 p^j_\alpha$, $p^i_\alpha = r_{\alpha,i} \leq^0 r'_{\alpha,i} \leq^0 p^{i+1}_\alpha$, and
(ii) $p^{i+1}_\alpha \Vdash$ " $r'_{\alpha,i}$ is the answer by $\underset{\sim}{\mathbf{st}}$ at stage i of the subgame ",
(iii) $\beta_{\alpha,i}$ satisfies the demand in $(\boxdot)_3$ and $(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) \in \#(r'_{\alpha,i})$,
(iv) $(\forall A \in d_{\alpha,i})(\forall \gamma \leq \alpha)(\exists \delta \in C^{p_\gamma})(A \cap Z^{p_\gamma}_\delta \in d^{p_\gamma}_\delta)$.

Then $p_{\alpha+1}$ is any $\leq^0$–upper bound to $\{p^i_\alpha : i < i_\alpha\}$.

The limit stages of the construction should be clear.

After the construction is carried out and we have $\bar{\sigma}_\lambda = \bigcup\{\bar{\sigma}_\alpha : \alpha < \lambda\}$, we define $r \in \mathbb{Q}^0_\lambda$ like r_ξ in the proof of 1.15 (see $(\odot)^9_{\xi,\alpha} + (\odot)^{10}_\xi$ there). Then r is $\leq^0$–stronger then all p_α (for $\alpha < \lambda$) and

$$r \Vdash_{\mathbb{Q}^0_\lambda} \text{" } \bar{\sigma}_\lambda \text{ is a play of } \Game^{\boxplus}(\Gamma_{\mathbb{Q}^0_\lambda}) \text{ in which INC uses } \underset{\sim}{\mathbf{st}} \text{ but COM wins ".}$$

(Note that the respective version of $(\odot)^{11}_\xi$ of the proof of 1.15 holds. By completeness it continues to hold in $\mathbf{V}^{\mathbb{Q}^0_\lambda}$.) $\square$

2. More on reasonably complete forcing

Definition 2.1. Let $\mathbb{P}$ be a forcing notion.

(1) For a condition $r \in \mathbb{P}$ let $\Game^\lambda_0(\mathbb{P}, r)$ be the following game of two players, *Complete* and *Incomplete*:

> the game lasts at most λ moves and during a play the players construct a sequence $\langle (p_i, q_i) : i < \lambda\rangle$ of pairs of conditions from $\mathbb{P}$ in such a way that $(\forall j < i < \lambda)(r \leq p_j \leq q_j \leq p_i)$ and at the stage $i < \lambda$ of the game, first Incomplete chooses p_i and then Complete chooses q_i.

Complete wins if and only if for every $i < \lambda$ there are legal moves for both players.

(2) We say that the forcing notion $\mathbb{P}$ is *strategically* $({<}\lambda)$*–complete* if Complete has a winning strategy in the game $\Game^\lambda_0(\mathbb{P}, p)$ for each condition $p \in \mathbb{P}$.

(3) Let $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ be a model such that ${}^{<\lambda}N \subseteq N$, $|N| = \lambda$ and $\mathbb{P} \in N$. We say that a condition $p \in \mathbb{P}$ is $(N, \mathbb{P})$*–generic in the standard sense* (or just: $(N, \mathbb{P})$*–generic*) if for every $\mathbb{P}$–name $\underset{\sim}{\tau} \in N$ for an ordinal we have $p \Vdash$ " $\underset{\sim}{\tau} \in N$ ".

(4) $\mathbb{P}$ is λ*–proper in the standard sense* (or just: λ*–proper*) if there is $x \in \mathcal{H}(\chi)$ such that for every model $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ satisfying

$$^{<\lambda}N \subseteq N, \quad |N| = \lambda \quad \text{and} \quad \mathbb{P}, x \in N,$$

and every condition $p \in N \cap \mathbb{P}$ there is an $(N, \mathbb{P})$–generic condition $q \in \mathbb{P}$ stronger than p.

Theorem 2.2 (See Shelah [9, Ch. III, Thm 4.1], Abraham [1, §2] and Eisworth [2, §3])**.** *Assume* $2^\lambda = \lambda^+$, $\lambda^{<\lambda} = \lambda$. *Let* $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \lambda^{++}\rangle$ *be* λ*–support iteration such that for all* $i < \lambda^{++}$ *we have*

- $\mathbb{P}_i$ *is* λ*–proper,*
- $\Vdash_{\mathbb{P}_i}$ *"*$|\mathbb{Q}_i| \leq \lambda^+$*".*

Then

(1) *for every* $\delta < \lambda^{++}$, $\Vdash_{\mathbb{P}_\delta} 2^\lambda = \lambda^+$, *and*
(2) *the limit* $\mathbb{P}_{\lambda^{++}}$ *satisfies the* λ^{++}*–cc.*

Proposition 2.3 ([7, Prop. A.1.6])**.** *Suppose* $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \mathbb{Q}_i : i < \gamma \rangle$ *is a* λ*–support iteration and, for each* $i < \gamma$,

$$\Vdash_{\mathbb{P}_i} \text{" } \mathbb{Q}_i \text{ is strategically } ({<}\lambda)\text{–complete ".}$$

Then, for each $\varepsilon \leq \gamma$ *and* $r \in \mathbb{P}_\varepsilon$, *there is a winning strategy* $\mathbf{st}(\varepsilon, r)$ *of Complete in the game* $\Game^\lambda_0(\mathbb{P}_\varepsilon, r)$ *such that, whenever* $\varepsilon_0 < \varepsilon_1 \leq \gamma$ *and* $r \in \mathbb{P}_{\varepsilon_1}$, *we have:*

(i) *if* $\langle (p_i, q_i) : i < \lambda \rangle$ *is a play of* $\Game^\lambda_0(\mathbb{P}_{\varepsilon_0}, r{\restriction}\varepsilon_0)$ *in which Complete follows the strategy* $\mathbf{st}(\varepsilon_0, r{\restriction}\varepsilon_0)$, *then* $\langle (p_i{}^\frown r{\restriction}[\varepsilon_0, \varepsilon_1), q_i{}^\frown r{\restriction}[\varepsilon_0, \varepsilon_1)) : i < \lambda \rangle$ *is a play of* $\Game^\lambda_0(\mathbb{P}_{\varepsilon_1}, r)$ *in which Complete uses* $\mathbf{st}(\varepsilon_1, r)$*;*
(ii) *if* $\langle (p_i, q_i) : i < \lambda \rangle$ *is a play of* $\Game^\lambda_0(\mathbb{P}_{\varepsilon_1}, r)$ *in which Complete plays according to the strategy* $\mathbf{st}(\varepsilon_1, r)$, *then* $\langle (p_i{\restriction}\varepsilon_0, q_i{\restriction}\varepsilon_0) : i < \lambda \rangle$ *is a play of* $\Game^\lambda_0(\mathbb{P}_{\varepsilon_0}, r{\restriction}\varepsilon_0)$ *in which Complete uses* $\mathbf{st}(\varepsilon_0, r{\restriction}\varepsilon_0)$*;*
(iii) *if* $\langle (p_i, q_i) : i < i^* \rangle$ *is a partial play of* $\Game^\lambda_0(\mathbb{P}_{\varepsilon_1}, r)$ *in which Complete uses* $\mathbf{st}(\varepsilon_1, r)$ *and* $p' \in \mathbb{P}_{\varepsilon_0}$ *is stronger than all* $p_i{\restriction}\varepsilon_0$ *(for* $i < i^*$*), then there is* $p^* \in \mathbb{P}_{\varepsilon_1}$ *such that* $p' = p^*{\restriction}\varepsilon_0$ *and* $p^* \geq p_i$ *for* $i < i^*$.

Definition 2.4 (Compare [6, Def. 1.4])**.** (1) Let γ be an ordinal, $w \subseteq \gamma$. *A standard* $(w,1)^\gamma$*–tree* is a pair $\mathcal{T} = (T, \mathrm{rk})$ such that
- $\mathrm{rk} : T \longrightarrow w \cup \{\gamma\}$,
- if $t \in T$ and $\mathrm{rk}(t) = \varepsilon$, then t is a sequence $\langle (t)_\zeta : \zeta \in w \cap \varepsilon \rangle$,
- $(T, \lhd)$ is a tree with root $\langle\rangle$ and such that every chain in T has a $\lhd$–upper bound it T,
- if $t \in T$, then there is $t' \in T$ such that $t \trianglelefteq t'$ and $\mathrm{rk}(t') = \gamma$.

We will keep the convention that $\mathcal{T}^x_y$ is (T^x_y, rk^x_y).

(2) Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \mathbb{Q}_i : i < \gamma \rangle$ be a λ–support iteration. *A standard tree of conditions in* $\bar{\mathbb{Q}}$ is a system $\bar{p} = \langle p_t : t \in T \rangle$ such that
- (T, rk) is a standard $(w,1)^\gamma$–tree for some $w \subseteq \gamma$, and
- $p_t \in \mathbb{P}_{\mathrm{rk}(t)}$ for $t \in T$, and
- if $s, t \in T$, $s \lhd t$, then $p_s = p_t{\restriction}\mathrm{rk}(s)$.

(3) Let $\bar{p}^0, \bar{p}^1$ be standard trees of conditions in $\bar{\mathbb{Q}}$, $\bar{p}^i = \langle p^i_t : t \in T \rangle$. We write $\bar{p}^0 \leq \bar{p}^1$ whenever for each $t \in T$ we have $p^0_t \leq p^1_t$.

Note that our standard trees and trees of conditions are a special case of that introduced in [7, Def. A.1.7] when $\alpha = 1$.

Proposition 2.5 (See [7, Prop. A.1.9])**.** *Assume that* $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \mathbb{Q}_i : i < \gamma \rangle$ *is a* λ*–support iteration such that for all* $i < \gamma$ *we have*

$$\Vdash_{\mathbb{P}_i} \text{" } \mathbb{Q}_i \text{ is strategically } ({<}\lambda)\text{–complete ".}$$

Suppose that $\bar{p} = \langle p_t : t \in T \rangle$ *is a standard tree of conditions in* $\bar{\mathbb{Q}}$, $|T| < \lambda$, *and* $\mathcal{I} \subseteq \mathbb{P}_\gamma$ *is open dense. Then there is a standard tree of conditions* $\bar{q} = \langle q_t : t \in T \rangle$ *such that* $\bar{p} \leq \bar{q}$ *and* $(\forall t \in T)(\mathrm{rk}(t) = \gamma \ \Rightarrow\ q_t \in \mathcal{I})$, *and such that conditions* q_{t_0}, q_{t_1} *are incompatible whenever* $t_0, t_1 \in T$, $\mathrm{rk}(t_0) = \mathrm{rk}(t_1)$ *but* $t_0 \neq t_1$.

Definition 2.6 (See [6, Def. 2.2])**.** Let $\mathbb{Q}$ be a forcing notion and let $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$ be a sequence of regular cardinals such that $\aleph_0 \leq \mu_\alpha \leq \lambda$ for all $\alpha < \lambda$.

(1) For a condition $p \in \mathbb{Q}$ we define a game $\Game^{\rm rcA}_{\bar{\mu}}(p, \mathbb{Q})$ between two players, Generic and Antigeneric, as follows. A play of $\Game^{\rm rcA}_{\bar{\mu}}(p, \mathbb{Q})$ lasts λ steps and during a play a sequence

$$\Big\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha \rangle : \alpha < \lambda \Big\rangle$$

is constructed. Suppose that the players have arrived to a stage $\alpha < \lambda$ of the game. Now,

$(\aleph)_\alpha$ first Generic chooses a non-empty set I_α of cardinality $< \mu_\alpha$ and a system $\langle p^\alpha_t : t \in I_\alpha \rangle$ of conditions from $\mathbb{Q}$,

$(\beth)_\alpha$ then Antigeneric answers by picking a system $\langle q^\alpha_t : t \in I_\alpha \rangle$ of conditions from $\mathbb{Q}$ such that $(\forall t \in I_\alpha)(p^\alpha_t \leq q^\alpha_t)$.

At the end, Generic wins the play

$$\Big\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha \rangle : \alpha < \lambda \Big\rangle$$

of $\Game^{\rm rcA}_{\bar{\mu}}(p, \mathbb{Q})$ if and only if

$(\circledast)^{\rm rc}_{\rm A}$ there is a condition $p^* \in \mathbb{Q}$ stronger than p and such that

$$p^* \Vdash_{\mathbb{Q}} \text{“ } \big(\forall \alpha < \lambda\big)\big(\exists t \in I_\alpha\big)\big(q^\alpha_t \in \Gamma_{\mathbb{Q}}\big) \text{ ”}.$$

(2) We say that a forcing notion $\mathbb{Q}$ is *reasonably A–bounding over* $\bar{\mu}$ if
  (a) $\mathbb{Q}$ is strategically $(<\lambda)$–complete, and
  (b) for any $p \in \mathbb{Q}$, Generic has a winning strategy in the game $\Game^{\rm rcA}_{\bar{\mu}}(p, \mathbb{Q})$.

Definition 2.7 (See [6, Def. 2.7])**.** Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underaccent{\tilde}{\mathbb{Q}}_\xi : \xi < \gamma \rangle$ be a λ–support iteration and let $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$ be a sequence of regular cardinals such that $\aleph_0 \leq \mu_\alpha \leq \lambda$ for all $\alpha < \lambda$.

(1) For a condition $p \in \mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ we define a game $\Game^{\rm tree\mathbf{A}}_{\bar{\mu}}(p, \bar{\mathbb{Q}})$ between two players, Generic and Antigeneric, as follows. A play of $\Game^{\rm tree\mathbf{A}}_{\bar{\mu}}(p, \bar{\mathbb{Q}})$ lasts λ steps and in the course of a play a sequence $\langle \mathcal{T}_\alpha, \bar{p}^\alpha, \bar{q}^\alpha : \alpha < \lambda \rangle$ is constructed. Suppose that the players have arrived to a stage $\alpha < \lambda$ of the game. Now,

$(\aleph)_\alpha$ first Generic chooses a standard $(w, 1)^\gamma$–tree $\mathcal{T}_\alpha$ such that $|T_\alpha| < \mu_\alpha$ and a tree of conditions $\bar{p}^\alpha = \langle p^\alpha_t : t \in T_\alpha \rangle \subseteq \mathbb{P}_\gamma$,

$(\beth)_\alpha$ then Antigeneric answers by picking a tree of conditions $\bar{q}^\alpha = \langle q^\alpha_t : t \in T_\alpha \rangle \subseteq \mathbb{P}_\gamma$ such that $\bar{p}^\alpha \leq \bar{q}^\alpha$.

At the end, Generic wins the play $\langle \mathcal{T}_\alpha, \bar{p}^\alpha, \bar{q}^\alpha : \alpha < \lambda \rangle$ of $\Game^{\rm tree\mathbf{A}}_{\bar{\mu}}(p, \bar{\mathbb{Q}})$ if and only if

$(\circledast)^{\rm tree}_{\mathbf{A}}$ there is a condition $p^* \in \mathbb{P}_\gamma$ stronger than p and such that

$$p^* \Vdash_{\mathbb{P}_\gamma} \text{“ } \big(\forall \alpha < \lambda\big)\big(\exists t \in T_\alpha\big)\big({\rm rk}_\alpha(t) = \gamma \ \&\ q^\alpha_t \in \Gamma_{\mathbb{P}_\gamma}\big) \text{ ”}$$

(2) We say that $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ is *reasonably* $A(\bar{\mathbb{Q}})$–bounding over* $\bar{\mu}$ if Generic has a winning strategy in the game $\Game^{\rm tree\mathbf{A}}_{\bar{\mu}}(p, \bar{\mathbb{Q}})$ for every $p \in \mathbb{P}_\gamma$.

Theorem 2.8 (See [6, Thm 2.8])**.** *Assume that*

(a) λ *is a strongly inaccessible cardinal,*

(b) $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$, *each* μ_α *is a regular cardinal satisfying (for* $\alpha < \lambda$*)*

$$\aleph_0 \le \mu_\alpha \le \lambda \qquad \textit{and} \qquad \big(\forall f \in {}^{\alpha}\mu_\alpha\big)\big(\big|\prod_{\xi<\alpha} f(\xi)\big| < \mu_\alpha\big),$$

(c) $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathop{\mathbb{Q}}\limits_{\sim}{}_\xi : \xi < \gamma \rangle$ *is a* λ*–support iteration such that for every* $\xi < \gamma$,

$$\Vdash_{\mathbb{P}_\xi} \text{“ } \mathop{\mathbb{Q}}\limits_{\sim}{}_\xi \textit{ is reasonably } A\textit{–bounding over } \bar{\mu} \text{ ”}.$$

Then $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ *is reasonably** $A(\bar{\mathbb{Q}})$*–bounding over* $\bar{\mu}$ *(and so* $\mathbb{P}_\gamma$ *is also* λ*–proper).*

Definition 2.9. Let $\mathbb{Q}$ be a forcing notion and let $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$ be a sequence of cardinals such that $\aleph_0 \le \mu_\alpha \le \lambda$ for all $\alpha < \lambda$. Suppose also that $\mathcal{U}$ is a normal filter on λ.

(1) For a condition $p \in \mathbb{Q}$ we define a game $\Game^{\mathrm{rc}\mathbf{2a}}_{\bar{\mu}}(p, \mathbb{Q})$ between Generic and Antigeneric as follows. A play of $\Game^{\mathrm{rc}\mathbf{2a}}_{\bar{\mu}}(p, \mathbb{Q})$ lasts at most λ steps and in the course of the play the players construct a sequence
($\boxtimes$) $\big\langle \xi_\alpha, \langle p^\alpha_\gamma, q^\alpha_\gamma : \gamma < \mu_\alpha \cdot \xi_\alpha \rangle : \alpha < \lambda \big\rangle$.
(Here μ_α is treated as an ordinal and $\mu_\alpha \cdot \xi_\alpha$ is the ordinal product of μ_α and ξ_α.) Suppose that the players have arrived to a stage $\alpha < \lambda$ of the game. First, Antigeneric picks a non-zero ordinal $\xi_\alpha < \lambda$. Then the two players start a subgame of length $\mu_\alpha \cdot \xi_\alpha$ alternately choosing the terms of the sequence $\langle p^\alpha_\gamma, q^\alpha_\gamma : \gamma < \mu_\alpha \cdot \xi_\alpha \rangle$. At a stage $\gamma = \mu_\alpha \cdot i + j$ (where $i < \xi_\alpha$, $j < \mu_\alpha$) of the subgame, first Generic picks a condition $p^\alpha_\gamma \in \mathbb{Q}$ stronger than p and stronger than all conditions q^α_δ for $\delta < \gamma$ of the form $\delta = \mu_\alpha \cdot i' + j$ (where $i' < i$), and then Antigeneric answers with a condition q^α_γ stronger than p^α_γ.
At the end, Generic wins the play ($\boxtimes$) of $\Game^{\mathrm{rc}\mathbf{2a}}_{\bar{\mu}}(p, \mathbb{Q})$ if and only if both players had always legal moves and
$(\circledast)^{\mathrm{rc}}_{\mathbf{2a}}$ there is a condition $p^* \in \mathbb{Q}$ stronger than p and such that

$$p^* \Vdash_{\mathbb{Q}} \text{“ } \big(\forall \alpha < \lambda\big)\big(\exists j < \mu_\alpha\big)\big(\{q^\alpha_{\mu_\alpha \cdot i + j} : i < \xi_\alpha\} \subseteq \Gamma_{\mathbb{Q}}\big) \text{ ”}.$$

(2) Games $\Game^{\mathrm{rc}\mathbf{2b}}_{\bar{\mu},\mathcal{U}}(p, \mathbb{Q})$ (for $p \in \mathbb{Q}$) are defined similarly, we only replace condition $(\circledast)^{\mathrm{rc}}_{\mathbf{2a}}$ by
$(\circledast)^{\mathrm{rc}}_{\mathbf{2b}}$ there is a condition $p^* \in \mathbb{Q}$ stronger than p and such that

$$p^* \Vdash_{\mathbb{Q}} \text{“ } \big\{\alpha < \lambda : \big(\exists j < \mu_\alpha\big)\big(\{q^\alpha_{\mu_\alpha \cdot i + j} : i < \xi_\alpha\} \subseteq \Gamma_{\mathbb{Q}}\big)\big\} \in \mathcal{U}^{\mathbb{Q}} \text{ ”},$$

where $\mathcal{U}^{\mathbb{Q}}$ is the ($\mathbb{Q}$–name for the) normal filter generated by $\mathcal{U}$ in $\mathbf{V}^{\mathbb{Q}}$.

(3) A strategy **st** for Generic in $\Game^{\mathrm{rc}\mathbf{2a}}_{\bar{\mu}}(p, \mathbb{Q})$ (or $\Game^{\mathrm{rc}\mathbf{2b}}_{\bar{\mu},\mathcal{U}}(p, \mathbb{Q})$) is said to be *nice* if for every play $\big\langle \xi_\alpha, \langle p^\alpha_\gamma, q^\alpha_\gamma : \gamma < \mu_\alpha \cdot \xi_\alpha \rangle : \alpha < \lambda \big\rangle$ in which she uses **st**, for every $\alpha < \lambda$, the conditions in $\{p^\alpha_\gamma : \gamma < \mu_\alpha\}$ are pairwise incompatible.

(4) Let $\mathbf{x} \in \{\mathbf{a}, \mathbf{b}\}$. A forcing notion $\mathbb{Q}$ is *nicely double* $\mathbf{x}$*–bounding over* $\bar{\mu}$ (and $\mathcal{U}$ if $\mathbf{x} = \mathbf{b}$) if
 (a) $\mathbb{Q}$ is strategically $(<\lambda)$–complete, and
 (b) Generic has a nice winning strategy in the game $\Game^{\mathrm{rc}\mathbf{2a}}_{\bar{\mu}}(p, \mathbb{Q})$ ($\Game^{\mathrm{rc}\mathbf{2b}}_{\bar{\mu},\mathcal{U}}(p, \mathbb{Q})$ if $\mathbf{x} = \mathbf{b}$) for every $p \in \mathbb{Q}$.

Definition 2.10 (See [6, Def. 5.1])**.** Suppose that λ is inaccessible and $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$ is an increasing sequence of cardinals below λ. We define a forcing notion

$\mathbb{P}^{\bar{\mu}}$ as follows.
A condition in $\mathbb{P}^{\bar{\mu}}$ is a pair $p = (f^p, C^p)$ such that

$$C^p \subseteq \lambda \text{ is a club of } \lambda \text{ and } f^p \in \prod\{\mu_\iota : \iota \in \lambda \setminus C^p\}.$$

The order $\leq_{\mathbb{P}^{\bar{\mu}}} = \leq$ **of** $\mathbb{P}^{\bar{\mu}}$ is given by:
$p \leq_{\mathbb{P}^{\bar{\mu}}} q$ if and only if $C^q \subseteq C^p$ and $f^p \subseteq f^q$.

Observation 2.11 (Compare [6, Prop. 5.2]). *Assume that $\bar{\mu}, \lambda$ are as in 2.10 above. Then the forcing notion $\mathbb{P}^{\bar{\mu}}$ is nicely double* **b***–bounding over $\bar{\mu}, \mathcal{D}_\lambda$.*

Theorem 2.12. *Assume that*

(a) *λ is a strongly inaccessible cardinal,*
(b) *$\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$ is a sequence of cardinals below λ such that $(\forall \alpha < \lambda)(\aleph_0 \leq \mu_\alpha = \mu_\alpha^{|\alpha|})$,*
(c) *$\bar{\mathbb{Q}} = \langle \mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\zeta : \zeta < \zeta^* \rangle$ is a λ–support iteration such that for every $\zeta < \gamma$,*

$$\Vdash_{\mathbb{P}_\zeta} \text{“ } \underset{\sim}{\mathbb{Q}}_\zeta \text{ is nicely double } \mathbf{a}\text{–bounding over } \bar{\mu} \text{ ”}.$$

Then $\mathbb{P}_{\zeta^*} = \lim(\bar{\mathbb{Q}})$ is nicely double **a***–bounding over $\bar{\mu}$ (and so $\mathbb{P}_{\zeta^*}$ is also λ–proper).*

Proof. For each $\zeta < \zeta^*$ pick a $\mathbb{P}_\zeta$–name $\underset{\sim}{\mathbf{st}}^0_\zeta$ such that

$$\Vdash_{\mathbb{P}_\zeta} \text{“ } \underset{\sim}{\mathbf{st}}^0_\zeta \text{ is a winning strategy for Complete in } \Game^\lambda_0\big(\underset{\sim}{\mathbb{Q}}_\zeta, \underset{\sim}{\emptyset}_{\underset{\sim}{\mathbb{Q}}_\zeta}\big) \text{ such that if Incomplete plays } \underset{\sim}{\emptyset}_{\underset{\sim}{\mathbb{Q}}_\zeta} \text{ then Complete answers with } \underset{\sim}{\emptyset}_{\underset{\sim}{\mathbb{Q}}_\zeta} \text{ as well ”}.$$

Let $p \in \mathbb{P}_{\zeta^*}$. We will describe a strategy $\mathbf{st}$ for Generic in the game $\Game^{\mathrm{rc}\mathbf{2a}}_{\bar{\mu}}(p, \mathbb{P}_{\zeta^*})$. In the course of a play, at a stage $\delta < \lambda$, Generic will be instructed to construct aside

$(\otimes)_\delta$ $w_\delta, \bar{t}^\delta, \xi^*_\delta$, $\underset{\sim}{\mathbf{st}}_\xi$ (for $\xi \in w_{\delta+1} \setminus w_\delta$), $\underset{\sim}{\bar{p}}_{\delta,\zeta}, \underset{\sim}{\bar{q}}_{\delta,\zeta}$, $p^{\delta,*}_\varepsilon$ (for $\varepsilon < \mu_\delta \cdot \xi_\delta$), and r^-_δ, r_δ.

These objects will be chosen so that if

$$\big\langle \xi_\delta, \langle p^\delta_\gamma, q^\delta_\gamma : \gamma < \mu_\delta \cdot \xi_\delta \rangle : \delta < \lambda \big\rangle$$

is a play of $\Game^{\mathrm{rc}\mathbf{2a}}_{\bar{\mu}}(p, \mathbb{P}_{\zeta^*})$ in which Generic follows $\mathbf{st}$, and the additional objects constructed at stage $\delta < \lambda$ are listed in $(\otimes)_\delta$, then the following conditions are satisfied (for each $\delta < \lambda$).

$(\boxtimes)_1$ $r^-_\delta, r_\delta \in \mathbb{P}_{\zeta^*}$, $r^-_0(0) = r_0(0) = p(0)$, $w_\delta \subseteq \zeta^*$, $|w_\delta| = |\delta + 1|$, $\bigcup\limits_{\alpha<\lambda} \mathrm{Dom}(r_\alpha) = \bigcup\limits_{\alpha<\lambda} w_\alpha$, $w_0 = \{0\}$, $w_\delta \subseteq w_{\delta+1}$ and if δ is limit then $w_\delta = \bigcup\limits_{\alpha<\delta} w_\alpha$.

$(\boxtimes)_2$ For each $\alpha < \delta <$ we have $(\forall \zeta \in w_{\alpha+1})(r_\alpha(\zeta) = r^-_\delta(\zeta) = r_\delta(\zeta))$ and $p \leq r^-_\alpha \leq r_\alpha \leq r^-_\delta \leq r_\delta$, and $p^{\delta,*}_\varepsilon \in \mathbb{P}_{\zeta^*}$ (for $\varepsilon < \mu_\delta \cdot \xi_\delta$).

$(\boxtimes)_3$ If $\zeta \in \zeta^* \setminus w_\delta$, then

$$\Vdash_{\mathbb{P}_\zeta} \text{“ the sequence } \langle r^-_\alpha(\zeta), r_\alpha(\zeta) : \alpha \leq \delta \rangle \text{ is a legal partial play of } \Game^\lambda_0\big(\underset{\sim}{\mathbb{Q}}_\zeta, \underset{\sim}{\emptyset}_{\underset{\sim}{\mathbb{Q}}_\zeta}\big) \text{ in which Complete follows } \underset{\sim}{\mathbf{st}}^0_\zeta \text{ ”}$$

and if $\zeta \in w_{\delta+1} \setminus w_\delta$, then $\underset{\sim}{\mathbf{st}}_\zeta$ is a $\mathbb{P}_\zeta$–name for a nice winning strategy for Generic in $\Game^{\mathrm{rc}\mathbf{2a}}_{\bar{\mu}}(r_\delta(\zeta), \underset{\sim}{\mathbb{Q}}_\zeta)$. (And $\mathbf{st}_0$ is a suitable winning strategy of Generic in $\Game^{\mathrm{rc}\mathbf{2a}}_{\bar{\mu}}(p(0), \mathbb{Q}_0)$.)

$(\boxtimes)_4$ $\bar{t}^\delta = \langle t^\delta_j : j < \mu_\delta \rangle$ is an enumeration of $\prod\limits_{\zeta \in w_\delta} \mu_\delta$.

$(\boxtimes)_5$ $\xi^*_\delta = \mu_\delta\cdot\xi_\delta$ (the ordinal product) and $\bar{\underaccent{\tilde}{p}}_{\delta,\zeta} = \langle \underaccent{\tilde}{p}^\gamma_{\delta,\zeta} : \gamma < \mu_\delta\cdot\xi^*_\delta\rangle$ and $\bar{\underaccent{\tilde}{q}}_{\delta,\zeta} = \langle \underaccent{\tilde}{q}^\gamma_{\delta,\zeta} : \gamma < \mu_\delta\cdot\xi^*_\delta\rangle$ are $\mathbb{P}_\zeta$–names for sequences of conditions in $\underaccent{\tilde}{\mathbb{Q}}_\zeta$ of length $\mu_\delta\cdot\xi^*_\delta$ (for $\zeta\in\bigcup\limits_{\alpha<\lambda} w_\alpha$).

$(\boxtimes)_6$ If $\zeta\in w_{\beta+1}\setminus w_\beta$, $\beta<\delta$, then

$$\Vdash_{\mathbb{P}_\zeta} \text{“ } \langle \xi^*_\alpha, \langle \underaccent{\tilde}{p}^\gamma_{\alpha,\zeta}, \underaccent{\tilde}{q}^\gamma_{\alpha,\zeta} : \gamma<\mu_\alpha\cdot\xi^*_\alpha\rangle : \alpha\le\delta\rangle \text{ is a partial play of } \partial^{\rm rc\mathbf{2a}}_{\bar\mu}(r_\beta(\zeta), \underaccent{\tilde}{\mathbb{Q}}_\zeta) \text{ in which Generic uses } \underaccent{\tilde}{\mathbf{st}}_\zeta \text{ ”.}$$

$(\boxtimes)_7$ If $\varepsilon=\mu_\delta\cdot i+j$, $i<\xi_\delta$, $j<\mu_\delta$, then

$$\mathrm{Dom}(p^{\delta,*}_\varepsilon) = \mathrm{Dom}(p^\delta_\varepsilon) = w_\delta\cup\mathrm{Dom}(p)\cup\bigcup_{\alpha<\delta}\mathrm{Dom}(r_\alpha)\cup\bigcup_{\varepsilon'<\varepsilon}\mathrm{Dom}(q^\delta_{\varepsilon'}),$$

and for each $\zeta\in w_\delta\cup\{\zeta^*\}$ the condition $p^{\delta,*}_\varepsilon{\restriction}\zeta$ is an upper bound to

$$\{p{\restriction}\zeta\}\cup\{r_\alpha{\restriction}\zeta : \alpha<\delta\}\cup$$
$$\{q^\delta_{\varepsilon'}{\restriction}\zeta : \varepsilon'=\mu_\delta\cdot i'+j'<\varepsilon \ \&\ i'<\xi_\delta \ \&\ j'<\mu_\delta \ \&\ t^\delta_{j'}{\restriction}\zeta = t^\delta_j{\restriction}\zeta\}.$$

$(\boxtimes)_8$ If $j<\mu_\delta$, $i<\xi_\delta$, $\zeta\in w_\delta$, $(t^\delta_j)_\zeta=\beta$ and $\varepsilon=\mu_\delta\cdot i+j$, $\gamma=\mu_\delta\cdot\varepsilon+\beta$, then $p^{\delta,*}_\varepsilon(\zeta)=p^\delta_\varepsilon(\zeta)=\underaccent{\tilde}{p}^\gamma_{\delta,\zeta}$ and $q^\delta_\varepsilon{\restriction}\zeta\Vdash_{\mathbb{P}_\zeta} q^\delta_\varepsilon(\zeta)=\underaccent{\tilde}{q}^\gamma_{\delta,\zeta}$.

$(\boxtimes)_9$ For each $\zeta\in\zeta^*\setminus w_\delta$ and $t\in\prod\{\mu_\delta:\xi\in w_\delta\cap\zeta\}$ we have:

$$\Vdash_{\mathbb{P}_\zeta} \text{“ the sequence } \langle p^{\delta,*}_\varepsilon(\zeta), p^\delta_\varepsilon(\zeta) : \varepsilon=\mu_\delta\cdot i+j \ \&\ i<\xi_\delta \ \&\ j<\mu_\delta \ \&\ t\trianglelefteq t^\delta_j\rangle \text{ is a legal partial play of } \partial^\lambda_0(\underaccent{\tilde}{\mathbb{Q}}_\zeta, p(\zeta)) \text{ in which Complete follows } \underaccent{\tilde}{\mathbf{st}}^0_\zeta \text{ ”.}$$

$(\boxtimes)_{10}$ $\mathrm{Dom}(r^-_\delta)=\mathrm{Dom}(r_\delta)=\bigcup\{\mathrm{Dom}(q^\delta_\varepsilon) : \varepsilon<\mu_\delta\cdot\xi_\delta\}$ and if $\zeta\in\zeta^*\setminus w_\delta$, $t\in\prod\{\mu_\delta:\xi\in w_\delta\cap\zeta\}$, then

$\Vdash_{\mathbb{P}_\zeta}$ “ if the set

$$\{p(\zeta)\}\cup\{r_\alpha(\zeta):\alpha<\delta\}\cup$$
$$\{q^\delta_\varepsilon(\zeta) : \varepsilon=\mu_\delta\cdot i+j \ \&\ i<\xi_\delta \ \&\ j<\mu_\delta \ \&\ t\trianglelefteq t^\delta_j \ \&\ q^\delta_\varepsilon{\restriction}\zeta\in\Gamma_{\mathbb{P}_\zeta}\}$$

has an upper bound in $\underaccent{\tilde}{\mathbb{Q}}_\xi$, then $r^-_\delta(\zeta)$ is such an upper bound, otherwise $r^-_\delta(\zeta)$ is just an upper bound to $\{p(\zeta)\}\cup\{r_\alpha(\zeta):\alpha<\delta\}$ ”.

Assume that the two players arrived to stage δ of $\partial^{\rm rc\mathbf{2a}}_{\bar\mu}(p,\mathbb{P}_{\zeta^*})$ and $\big\langle \xi_\alpha, \langle p^\alpha_\varepsilon, q^\alpha_\varepsilon : \varepsilon<\mu_\alpha\cdot\xi_\alpha\rangle : \alpha<\delta\big\rangle$ is the play constructed so far, and that Generic followed $\mathbf{st}$ and determined objects listed in $(\otimes)_\alpha$ (for $\alpha<\delta$) with properties $(\boxtimes)_1$–$(\boxtimes)_{10}$.

Below, whenever we say *Generic chooses x such that* we mean *Generic chooses the $<^*_\chi$–first x such that*, etc.

First, Generic uses her favorite bookkeeping device to determine w_δ so that the demands of $(\boxtimes)_1$ are satisfied (and that at the end we will have $\bigcup\limits_{\alpha<\lambda}\mathrm{Dom}(r_\alpha) = \bigcup\limits_{\alpha<\lambda} w_\alpha$). If $\beta<\delta$ and $\zeta\in w_\beta$, then we have already $\bar{\underaccent{\tilde}{p}}_{\alpha,\zeta},\bar{\underaccent{\tilde}{q}}_{\alpha,\zeta}$ for $\alpha<\delta$ (see $(\boxtimes)_6$), but we have not yet defined those objects when $\delta=\delta_0+1$ and $\zeta\in w_\delta\setminus w_{\delta_0}$. So if $\delta=\delta_0+1$ and $\zeta\in w_\delta\setminus w_{\delta_0}$ then let $\bar{\underaccent{\tilde}{p}}_{\alpha,\zeta}=\langle\underaccent{\tilde}{p}^\gamma_{\alpha,\zeta}:\gamma<\mu_\alpha\cdot\xi^*_\alpha\rangle$ and $\bar{\underaccent{\tilde}{q}}_{\alpha,\zeta}=\langle\underaccent{\tilde}{q}^\gamma_{\alpha,\zeta}:\gamma<\mu_\alpha\cdot\xi^*_\alpha\rangle$ (for $\alpha<\delta$) be such that

$$\Vdash_{\mathbb{P}_\zeta} \text{“ } \langle \xi^*_\alpha, \langle \underaccent{\tilde}{p}^\gamma_{\alpha,\zeta}, \underaccent{\tilde}{q}^\gamma_{\alpha,\zeta} : \gamma<\mu_\alpha\cdot\xi^*_\alpha\rangle : \alpha<\delta\rangle \text{ is a partial play of } \partial^{\rm rc\mathbf{2a}}_{\bar\mu}(r_{\delta_0}(\zeta), \underaccent{\tilde}{\mathbb{Q}}_\zeta) \text{ in which Generic uses } \underaccent{\tilde}{\mathbf{st}}_\zeta \text{ and } \underaccent{\tilde}{p}^\gamma_{\alpha,\zeta}=\underaccent{\tilde}{q}^\gamma_{\alpha,\zeta} \text{ for all } \alpha<\delta,\ \gamma<\mu_\alpha\cdot\underaccent{\tilde}{\xi}^*_\alpha \text{ ”.}$$

Condition $(\boxtimes)_4$ and our rule of taking “the $<^*_\chi$–first” determine the enumeration $\bar t^\delta=\langle t^\delta_j : j<\mu_\delta\rangle$ of $\prod\limits_{\zeta\in w_\delta}\mu_\delta$. Now Antigeneric picks ξ_δ and the two players start a subgame of length $\mu_\delta\cdot\xi_\delta$. During the subgame Generic will simulate subgames

of level δ at coordinates $\zeta \in w_\delta$ pretending that Antigeneric played $\xi^*_\delta = \mu_\delta \cdot \xi_\delta$ there. Each step in the subgame of $\Game^{\rm rc\mathbf{2a}}_{\bar{\mu}}(p, \mathbb{P}_{\zeta^*})$ will correspond to μ_δ steps in the subgames of $\Game^{\rm rc\mathbf{2a}}_{\bar{\mu}}(r_\beta(\zeta), \mathop{\mathbb{Q}}\limits_{\sim}{}_\zeta)$ (when $\zeta \in w_{\beta+1} \setminus w_\beta$, $\beta < \delta$). So suppose that the two opponents have arrived to a stage $\varepsilon = \mu_\delta \cdot i + j$ of the subgame, $i < \xi_\delta$, $j < \mu_\delta$, and assume also that Generic (playing according to $\mathbf{st}$) has already defined $\mathop{p}\limits_{\sim}{}^\gamma_{\delta,\zeta}, \mathop{q}\limits_{\sim}{}^\gamma_{\delta,\zeta}$ for $\zeta \in w_\delta$, $\gamma < \mu_\delta \cdot \varepsilon$ so that the requirements of $(\boxtimes)_6 + (\boxtimes)_8$ are satisfied. For each $\zeta \in w_\delta$ and $\beta < (t^\delta_j)_\zeta$ let $\mathop{p}\limits_{\sim}{}^{\mu_\delta \cdot \varepsilon + \beta}_{\delta,\zeta} = \mathop{q}\limits_{\sim}{}^{\mu_\delta \cdot \varepsilon + \beta}_{\delta,\zeta}$ be $\mathbb{P}_\zeta$–names for conditions in $\mathop{\mathbb{Q}}\limits_{\sim}{}_\zeta$ such that (the relevant part of) $(\boxtimes)_6$ holds. The same clause determines also $\mathop{p}\limits_{\sim}{}^{\mu_\delta \cdot \varepsilon + \beta}_{\delta,\zeta}$ for $\beta = (t^\delta_j)_\zeta$, $\zeta \in w_\delta$. Then the requirements in $(\boxtimes)_7 + (\boxtimes)_8$ essentially describe what $p^{\delta,*}_\varepsilon$ is. Note that the "upper bound demands" in $(\boxtimes)_7$ can be satisfied because of $(\boxtimes)_9 + (\boxtimes)_{10}$. Next, Generic's inning p^δ_ε in $\Game^{\rm rc\mathbf{2a}}_{\bar{\mu}}(p, \mathbb{P}_{\zeta^*})$ is chosen so that $\mathrm{Dom}(p^\delta_\varepsilon) = \mathrm{Dom}(p^{\delta,*}_\varepsilon)$ and clauses $(\boxtimes)_8 + (\boxtimes)_9$ hold. After this Antigeneric answers with a condition $q^\delta_\varepsilon \geq p^\delta_\varepsilon$, and Generic picks for the construction aside names $\mathop{q}\limits_{\sim}{}^{\mu_\delta \cdot \varepsilon + \beta}_{\delta,\zeta}$ for $\zeta \in w_\delta$ by the demand in $(\boxtimes)_8$. She also picks $\mathop{p}\limits_{\sim}{}^{\mu_\delta \cdot \varepsilon + \beta}_{\delta,\zeta} = \mathop{q}\limits_{\sim}{}^{\mu_\delta \cdot \varepsilon + \beta}_{\delta,\zeta}$ for $\zeta \in w_\delta$ and $(t^\delta_j)_\zeta < \beta < \mu_\delta$ so that $(\boxtimes)_6$ holds.

This completes the description of what happens during the $\mu_\delta \cdot \xi_\delta$ steps of the subgame. After the subgame is over and the sequence $\langle p^\delta_\gamma, q^\delta_\gamma : \gamma < \mu_\delta \cdot \xi_\delta \rangle$ is constructed, Generic chooses conditions $r^-_\delta, r_\delta \in \mathbb{P}_{\zeta^*}$ by $(\boxtimes)_1 + (\boxtimes)_2 + (\boxtimes)_3 + (\boxtimes)_{10}$. (Note: since $\mathop{\mathbf{st}}\limits_{\sim}{}_\zeta$ are names for nice strategies, if $\zeta \in \zeta^* \setminus w_\delta$, $i_0, i_1 < \xi_\delta$, $j_0, j_1 < \mu_\delta$, $\varepsilon_0 = \mu_\delta \cdot i_0 + j_0$, $\varepsilon_1 = \mu_\delta \cdot i_1 + j_1$, $t_0, t_1 \in \prod\{\mu_\delta : \xi \in w_\delta \cap \zeta\}$, $t_0 \trianglelefteq t^\delta_{j_0}$, $t_1 \trianglelefteq t^\delta_{j_1}$ and $t_0 \neq t_1$, then the conditions $q^\delta_{\varepsilon_0}{\restriction}\zeta, q^\delta_{\varepsilon_1}{\restriction}\zeta$ are incompatible.)

This finishes the description of the strategy $\mathbf{st}$.

Let us argue that $\mathbf{st}$ is a winning strategy for Generic. Suppose that

$$\big\langle \xi_\delta, \langle p^\delta_\gamma, q^\delta_\gamma : \gamma < \mu_\delta \cdot \xi_\delta \rangle : \delta < \lambda \big\rangle$$

is a play of $\Game^{\rm rc\mathbf{2a}}_{\bar{\mu}}(p, \mathbb{P}_{\zeta^*})$ in which Generic followed $\mathbf{st}$ and she constructed the side objects listed in $(\otimes)_\delta$ (for $\delta < \lambda$) so that demands $(\boxtimes)_1$–$(\boxtimes)_{10}$ are satisfied. We define a condition $r \in \mathbb{P}_{\zeta^*}$ as follows. Let $\mathrm{Dom}(r) = \bigcup\limits_{\delta<\lambda} \mathrm{Dom}(r_\delta)$. For $\zeta \in \mathrm{Dom}(r)$ let $r(\zeta)$ be a $\mathbb{P}_\zeta$–name for a condition in $\mathop{\mathbb{Q}}\limits_{\sim}{}_\zeta$ such that

$(\boxtimes)_{11}$ if $\zeta \in w_{\alpha+1} \setminus w_\alpha$, $\alpha < \lambda$ (or $\zeta = \alpha = 0$), then

$$\Vdash_{\mathbb{P}_\zeta} \text{“ } r(\zeta) \geq r_\alpha(\zeta) \text{ and } r(\zeta) \Vdash_{\mathop{\mathbb{Q}}\limits_{\sim}{}_\zeta} \big(\forall \delta < \lambda\big)\big(\exists j < \mu_\delta\big)\big(\forall \varepsilon < \xi^*_\delta\big)\big(\mathop{q}\limits_{\sim}{}^{\mu_\delta \cdot \varepsilon + j}_{\delta,\zeta} \in \Gamma_{\mathop{\mathbb{Q}}\limits_{\sim}{}_\zeta}\big) \text{ ”.}$$

Clearly r is well defined (remember $(\boxtimes)_6$) and $(\forall \delta < \lambda)(r_\delta \leq r)$ and $p \leq r$.

Suppose now that $\delta < \lambda$ and $r' \geq r$. We are going to find $j < \mu_\delta$ and a condition $r'' \geq r'$ such that $(\forall i < \xi_\delta)(q^\delta_{\mu_\delta \cdot i + j} \leq r'')$. To this end let $\langle \zeta_\alpha : \alpha \leq \alpha^* \rangle$ be the increasing enumeration of $w_\delta \cup \{\zeta^*\}$. For $\zeta \leq \zeta^*$ and $q \in \mathbb{P}_\zeta$, let $\mathbf{st}(\zeta, q)$ be a winning strategy of Complete in $\Game^\lambda_0(\mathbb{P}_\zeta, q)$ with the coherence properties given in 2.3.

By induction on $\alpha \leq \alpha^*$ we will choose conditions $r^*_\alpha, r^{**}_\alpha \in \mathbb{P}_{\zeta_\alpha}$ and $(t)_{\zeta_\alpha} < \mu_\delta$ such that

$(\boxtimes)_{12}$ $r'{\restriction}\zeta_\alpha \leq r^*_\alpha$,

$(\boxtimes)_{13}$ if $i < \xi_\delta$, $j < \mu_\delta$, $(t^\delta_j)_{\zeta_\beta} = (t)_{\zeta_\beta}$ for $\beta < \alpha$, then $q^\delta_{\mu_\delta \cdot i + j}{\restriction}\zeta_\alpha \leq r^*_\alpha$,

$(\boxtimes)_{14}$ $\langle r^*_\beta{}^\frown r'{\restriction}[\zeta_\beta, \zeta^*), r^{**}_\beta{}^\frown r'{\restriction}[\zeta_\beta, \zeta^*) : \beta < \alpha \rangle$ is a partial legal play of $\Game^\lambda_0(\mathbb{P}_{\zeta^*}, r')$ in which Complete uses her winning strategy $\mathbf{st}(\zeta^*, r')$.

Suppose that $\alpha \leq \alpha^*$ is a limit ordinal and we have already defined $(t)_{\zeta_\beta} < \mu_\delta$ and $r^*_\beta, r^{**}_\beta \in \mathbb{P}_{\zeta_\beta}$ for $\beta < \alpha$. Let $\zeta = \sup(\zeta_\beta : \beta < \alpha)$. It follows from $(\boxtimes)_{14}$ that we may pick a condition $s \in \mathbb{P}_\zeta$ stronger than all r^{**}_β for $\beta < \alpha$. Put $r^*_\alpha = s^\frown r'{\restriction}[\zeta, \zeta_\alpha) \in \mathbb{P}_{\zeta_\alpha}$. Then plainly $r'{\restriction}\zeta_\alpha \leq r^*_\alpha$ and $q^\delta_{\mu_\delta \cdot i+j}{\restriction}\zeta \leq r^*_\alpha{\restriction}\zeta$ whenever

$(\boxtimes)^{i,j,\alpha}_{15}$ $i < \xi_\delta$, $j < \mu_\delta$ and $(t^\delta_j)_{\zeta_\beta} = (t)_{\zeta_\beta}$ for all $\beta < \alpha$.

Now by induction on $\xi \leq \zeta_\alpha$ we show that $q^\delta_{\mu_\delta \cdot i+j}{\restriction}\xi \leq r^*_\alpha{\restriction}\xi$ whenever $(\boxtimes)^{i,j,\alpha}_{15}$ holds. For $\xi \leq \zeta$ we are already done, so assume $\xi \in [\zeta, \zeta_\alpha)$ and we have shown that $q^\delta_{\mu_\delta \cdot i+j}{\restriction}\xi \leq r^*_\alpha{\restriction}\xi$ whenever $(\boxtimes)^{i,j,\alpha}_{15}$ holds. It follows from $(\boxtimes)_7 + (\boxtimes)_9$ that

$$r^*_\alpha{\restriction}\xi \Vdash \text{“ the set } \{p(\xi)\} \cup \{r_\alpha(\xi) : \alpha < \delta\} \cup \{q^\delta_\varepsilon(\xi) : \varepsilon = \mu_\delta \cdot i + j \ \&\ i < \xi_\delta \ \&\ j < \mu_\delta \ \&\ (\forall \beta < \alpha)\big((t^\delta_j)_{\zeta_\beta} = (t)_{\zeta_\beta}\big)\} \text{ has an upper bound in } \underset{\sim}{\mathbb{Q}}_\xi \text{ ”.}$$

and therefore we may use $(\boxtimes)_{10}$ to conclude that

$$r^*_\alpha{\restriction}\xi \Vdash \text{“ if } (\boxtimes^{i,j,\alpha}_{15}) \text{ holds, then } q^\delta_{\mu_\delta \cdot i+j}(\xi) \leq r_\delta(\xi) \leq r'(\xi) = r^*_\alpha(\xi) \text{ ”.}$$

The limit stages are trivial and we may claim that $q^\delta_{\mu_\delta \cdot i+j}{\restriction}\zeta_\alpha \leq r^*_\alpha$ whenever $(\boxtimes^{i,j,\alpha}_{15})$ holds. Next, r^{**}_α is determined by $(\boxtimes)_{14}$.

Now suppose that $\alpha = \beta + 1 \leq \alpha^*$ and we have already defined $r^*_\beta, r^{**}_\beta \in \mathbb{P}_{\zeta_\beta}$ and $\langle (t)_{\zeta_\gamma} : \gamma < \beta \rangle$. It follows from $(\boxtimes)_{11}$ that

$$r^{**}_\beta \Vdash_{\mathbb{P}_{\zeta_\beta}} \text{“ } r(\zeta_\beta) \Vdash_{\underset{\sim}{\mathbb{Q}}_{\zeta_\beta}} \big(\exists \rho < \mu_\delta\big)\big(\forall \varepsilon < \xi^*_\delta\big)\big(\underset{\sim}{q}^{\mu_\delta \cdot \varepsilon + \rho}_{\delta, \zeta_\beta} \in \Gamma_{\underset{\sim}{\mathbb{Q}}_{\zeta_\beta}}\big) \text{ ”,}$$

so we may pick $\rho = (t)_{\zeta_\beta}$ and a condition $s \in \mathbb{P}_{\zeta_\beta + 1}$ such that $r^{**}_\beta \leq s{\restriction}\zeta_\beta$ and

$$s{\restriction}\zeta_\beta \Vdash_{\mathbb{P}_{\zeta_\beta}} \big(\forall \varepsilon < \xi^*_\delta\big)\big(\underset{\sim}{q}^{\mu_\delta \cdot \varepsilon + \rho}_{\delta, \zeta_\beta} \leq s(\zeta_\beta)\big).$$

It follows from $(\boxtimes)_{13}$+$(\boxtimes)_8$ that then also $q^\delta_{\mu_\delta \cdot i+j}{\restriction}(\zeta_\beta + 1) \leq s$ whenever $i < \xi_\delta$, $j < \mu_\delta$ and $(t^\delta_j)_{\zeta_\gamma} = (t)_{\zeta_\gamma}$ for $\gamma \leq \beta$. We let $r^*_\alpha = s^\frown r'{\restriction}(\zeta_\beta, \zeta_\alpha)$ and exactly like in the limit case we argue that $r'{\restriction}\zeta_\alpha \leq r^*_\alpha$ and $q^\delta_{\mu_\delta \cdot i+j}{\restriction}\zeta_\alpha \leq r^*_\alpha$ whenever $i < \xi_\delta$, $j < \mu_\delta$ and $(t^\delta_j)_{\zeta_\gamma} = (t)_{\zeta_\gamma}$ for $\gamma \leq \beta$. Again, r^{**}_α is determined by $(\boxtimes)_{14}$.

After the induction is completed look at $r'' = r^*_{\alpha^*}$ and $j < \mu_\delta$ such that $t^\delta_j = \langle (t)_{\zeta_\alpha} : \alpha < \alpha^* \rangle$. □

Theorem 2.13. *Assume (a), (b) of 2.12. Suppose that $\mathcal{U}$ is a normal filter on λ and*

(c) $\bar{\mathbb{Q}} = \langle \mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\zeta : \zeta < \zeta^* \rangle$ *is a λ–support iteration such that for every $\zeta < \gamma$,*

$$\Vdash_{\mathbb{P}_\zeta} \text{“ } \underset{\sim}{\mathbb{Q}}_\zeta \textit{ is nicely double } \mathbf{b}\textit{–bounding over } \bar{\mu}, \mathcal{U}^{\mathbb{P}_\zeta} \text{ ”.}$$

Then $\mathbb{P}_{\zeta^*} = \lim(\bar{\mathbb{Q}})$ is nicely double $\mathbf{b}$–bounding over $\bar{\mu}, \mathcal{U}$.

Proof. The proof essentially repeats that of 2.12 with arguments as in [6, Claim 2.5.1]. □

3. Reasonable ultrafilters with small generating systems

Our aim here is to show that, consistently, there may exist a very reasonable ultrafilter on an inaccessible cardinal λ with generating system of size less than 2^λ.

Lemma 3.1. *Assume that $G^* \subseteq \mathbb{Q}^0_\lambda$ is directed (with respect to $\leq^0$) and $\mathrm{fil}(G^*)$ is an ultrafilter on λ, $r \in G^*$. Let $\mathbb{P}$ be a forcing notion not adding bounded subsets of λ, $p \in \mathbb{P}$ and let $\underset{\sim}{A}$ be a $\mathbb{P}$–name for a subset of λ such that $p \Vdash_{\mathbb{P}} \underset{\sim}{A} \in \big(\mathrm{fil}(G^*)\big)^+$. Then*

$$Y \stackrel{\rm def}{=} \bigcup \big\{ Z^r_\delta : \delta \in C^r \text{ and } p \nVdash_{\mathbb{P}} \text{`` } \underset{\sim}{A} \cap Z^r_\delta \notin d^r_\delta \text{ ''}\big\} \in \mathrm{fil}(G^*).$$

Proof. Assume towards contradiction that $Y \notin \mathrm{fil}(G^*)$. Then we may find $s \in G^*$ such that $r \leq^0 s$ and $\lambda \setminus Y \in \mathrm{fil}(s)$. Take $\varepsilon < \lambda$ such that

if $\alpha \in C^s \setminus \varepsilon$,
then $Z^s_\alpha \setminus Y \in d^s_\alpha$ and $(\forall A \in d^s_\alpha)(\exists \beta \in C^r)(A \cap Z^r_\beta \in d^r_\beta)$.

(Remember 1.5.) Now take a generic filter $G \subseteq \mathbb{P}$ over $\mathbf{V}$ such that $p \in G$ and work in $\mathbf{V}[G]$. Since $\underset{\sim}{A}^G \in \mathrm{fil}(s)^+$, we may pick $\alpha \in C^s$ such that $\varepsilon < \alpha$ and $\underset{\sim}{A}^G \cap Z^s_\alpha \in d^s_\alpha$. Then also $Z^s_\alpha \cap \underset{\sim}{A}^G \setminus Y \in d^s_\alpha$ and thus we may find $\beta \in C^r$ such that $Z^s_\alpha \cap \underset{\sim}{A}^G \cap Z^r_\beta \setminus Y \in d^r_\beta$. In particular, $Z^r_\beta \setminus Y \neq \emptyset$, so $p \Vdash \underset{\sim}{A} \cap Z^r_\beta \notin d^r_\beta$, and thus $\underset{\sim}{A}^G \cap Z^r_\beta \notin d^r_\beta$. Consequently $Z^s_\alpha \cap \underset{\sim}{A}^G \cap Z^r_\beta \setminus Y \notin d^r_\beta$ giving a contradiction. □

Theorem 3.2. *Assume that*

(i) *λ is strongly inaccessible, $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda\rangle$, each μ_α is a regular cardinal, $\aleph_0 \leq \mu_\alpha \leq \lambda$ and $\big(\forall f \in {}^\alpha\mu_\alpha\big)\big(\big|\prod\limits_{\xi<\alpha} f(\xi)\big| < \mu_\alpha\big)$ for $\alpha < \lambda$;*
(ii) *$\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi : \xi < \gamma\rangle$ is a λ–support iteration such that for every $\xi < \gamma$,*
$$\Vdash_{\mathbb{P}_\xi} \text{`` } \underset{\sim}{\mathbb{Q}}_\xi \text{ is reasonably } A\text{–bounding over } \bar{\mu} \text{ '';}$$
(iii) *$G^* \subseteq \mathbb{Q}^0_\lambda$ is a $\leq^0$–downward closed $\bar{\mu}$–super reasonable family such that $\mathrm{fil}(G^*)$ is an ultrafilter on λ.*

Then
$$\Vdash_{\mathbb{P}_\gamma} \text{`` } \mathrm{fil}(G^*) \text{ is an ultrafilter on } \lambda \text{ ''.}$$

Proof. The proof is by induction on the length γ of the iteration $\bar{\mathbb{Q}}$. So we assume that (i)–(iii) hold and for each $\xi < \gamma$

$(\odot)_\xi$ $\Vdash_{\mathbb{P}_\xi}$ " $\mathrm{fil}(G^*)$ is an ultrafilter on λ ".

Note that (by the strategic $({<}\lambda)$–completeness of $\mathbb{P}_\gamma$) forcing with $\mathbb{P}_\gamma$ does not add bounded subsets of λ, and therefore $\big(\mathbb{Q}^0_\lambda\big)^{\mathbf{V}} \subseteq \big(\mathbb{Q}^0_\lambda\big)^{\mathbf{V}^{\mathbb{P}}}$.

Claim 3.2.1. *Assume that*

(a) *$\underset{\sim}{A}$ is a $\mathbb{P}_\gamma$–name for a subset of λ such that $\Vdash_{\mathbb{P}_\gamma} \underset{\sim}{A} \in \big(\mathrm{fil}(G^*)\big)^+$,*
(b) *$w \in [\lambda]^{<\omega}$ and $\mathcal{T}$ is a finite standard $(w,1)^\gamma$–tree, and*
(c) *$\bar{p} = \langle p_t : t \in T\rangle$ is a (finite) tree of conditions in $\bar{\mathbb{Q}}$, and*
(d) *$r \in G^*$ and X is the set of all $\alpha \in C^r$ for which there is a tree of conditions $\bar{q} = \langle q_t : t \in T\rangle$ such that $(\forall t \in T)(\mathrm{rk}(t) = \gamma \ \Rightarrow\ q_t \Vdash \underset{\sim}{A} \cap Z^r_\alpha \in d^r_\alpha)$ and $\bar{q} \geq \bar{p}$.*

Then $\bigcup\{Z^r_\alpha : \alpha \in X\} \in \mathrm{fil}(G^*)$.

Proof of the Claim. Induction on $|w|$.

If $w = \emptyset$ and so $T = \{\langle\rangle\}$, then the assertion follows directly from Lemma 3.1 (with $p, \mathbb{P}$ there standing for $p_{\langle\rangle}, \mathbb{P}_\gamma$ here).

Assume that $|w| = n+1$, $\xi^* = \max(w)$, $w' = w \setminus \{\xi^*\}$ and the claim is true for w' (in place of w) and any $\underset{\sim}{A}, \bar{p}$. Let $\underset{\sim}{\mathbb{P}}_{\xi^*\gamma}$ be a $\mathbb{P}_{\xi^*}$–name for a forcing notion with universe $P_{\xi^*\gamma} = \{p{\restriction}[\xi^*,\gamma) : p \in \mathbb{P}_\gamma\}$ such that

if $G\subseteq \mathbb{P}_{\xi^*}$ is generic over $\mathbf{V}$ and $f,g\in P_{\xi^*\gamma}$,
then $\mathbf{V}[G]\models f\leq_{\underset{\sim}{\mathbb{P}}_{\xi^*\gamma}[G]} g$ if and only if $(\exists p\in G)(p\cup f\leq_{\mathbb{P}_\gamma} p\cup g)$.

Note that $P_{\xi^*\gamma}$ is from $\mathbf{V}$ but the relation $\leq_{\underset{\sim}{\mathbb{P}}_{\xi^*\gamma}[G]}$ is defined in $\mathbf{V}[G]$ only. Also $\mathbb{P}_\gamma$ is isomorphic with a dense subset of the composition $\mathbb{P}_{\xi^*}*\underset{\sim}{\mathbb{P}}_{\xi^*\gamma}$.

We are going to define a $\mathbb{P}_{\xi^*}$–name $\underset{\sim}{Y}$ for a subset of λ. Suppose that $G\subseteq\mathbb{P}_{\xi^*}$ is generic over $\mathbf{V}$ and work in $\mathbf{V}[G]$. For $t\in T$ such that $\mathrm{rk}(t)=\gamma$ let X_t consist of all $\alpha\in C^r$ for which there is $f\in P_{\xi^*\gamma}$ such that

$$p_t\restriction[\xi^*,\gamma)\leq_{\underset{\sim}{\mathbb{P}}_{\xi^*\gamma}[G]} f\quad\text{ and }\quad f\Vdash_{\underset{\sim}{\mathbb{P}}_{\xi^*\gamma}[G]}\underset{\sim}{A}\cap Z^r_\alpha\in d^r_\alpha.$$

Let $Y_t=\bigcup\big\{Z^r_\alpha:\alpha\in X_t\big\}$ (for $t\in T$ such that $\mathrm{rk}(t)=\gamma$). It follows from Lemma 3.1 that each Y_t belongs to $\mathrm{fil}(G^*)$ (remember that $\Vdash_{\mathbb{P}_{\xi^*}}$ " $\mathrm{fil}(G^*)$ is an ultrafilter" by $(\odot)_{\xi^*}$). Hence

$$Y^*\stackrel{\rm def}{=}\bigcap\big\{Y_t:t\in T\ \&\ \mathrm{rk}(t)=\gamma\big\}\in\mathrm{fil}(G^*).$$

Note that for each $\alpha\in C^r$, either $Z^r_\alpha\cap Y^*=\emptyset$ or $Z^r_\alpha\subseteq Y^*$.

Going back to $\mathbf{V}$, let $\underset{\sim}{Y}^*,\underset{\sim}{Y}_t,\underset{\sim}{X}_t$ be $\mathbb{P}_{\xi^*}$–names for the objects described as Y^*,Y_t,X_t above. Thus $\Vdash_{\mathbb{P}_{\xi^*}}\underset{\sim}{Y}^*\in\mathrm{fil}(G^*)$ and we may apply the inductive hypothesis to w', $T'=\{t\restriction\xi^*:t\in T\}$ and $\bar p'=\{p_{t'}:t'\in T'\}$. Thus, if X^* is the set of all $\alpha\in C^r$ for which there is a tree of conditions $\bar q'=\langle q'_{t'}:t'\in T'\rangle\subseteq\mathbb{P}_{\xi^*}$ such that $\bar q'\geq\bar p'$ and

$$\big(\forall t'\in T\big)\big(\mathrm{rk}(t')=\xi^*\ \Rightarrow\ q'_{t'}\Vdash_{\mathbb{P}_{\xi^*}}\underset{\sim}{Y}^*\cap Z^r_\alpha\in d^r_\alpha\big),$$

then $\bigcup\big\{Z^r_\alpha:\alpha\in X^*\big\}\in\mathrm{fil}(G^*)$.

Now suppose that $\alpha\in X^*$ as witnessed by $\bar q'$ and let $t'\in T$ be such that $\mathrm{rk}(t')=\xi^*$ (so $\mathrm{rk}'(t')=\gamma$). Then $q'_{t'}\Vdash_{\mathbb{P}_{\xi^*}} Z^r_\alpha\subseteq\underset{\sim}{Y}^*$ and hence $q'_{t'}\Vdash_{\mathbb{P}_{\xi^*}}\alpha\in\underset{\sim}{X}_t$ for all $t\in T$ with $\mathrm{rk}(t)=\gamma$, so we have $\mathbb{P}_{\xi^*}$–names $\underset{\sim}{f}{}^{t'}_t$ for elements of $\underset{\sim}{\mathbb{P}}_{\xi^*\gamma}$ such that

$$q'_{t'}\Vdash_{\mathbb{P}_{\xi^*}}\text{" } p_t\restriction[\xi^*,\gamma)\leq_{\underset{\sim}{\mathbb{P}}_{\xi^*\gamma}}\underset{\sim}{f}{}^{t'}_t\ \&\ \underset{\sim}{f}{}^{t'}_t\Vdash_{\underset{\sim}{\mathbb{P}}_{\xi^*\gamma}}\underset{\sim}{A}\cap Z^r_\alpha\in d^r_\alpha\text{ ".}$$

Now use 2.5 (or just finite induction) to get a tree of conditions $\bar q''=\langle q''_{t'}:t'\in T'\rangle\subseteq\mathbb{P}_{\xi^*}$ and objects $g^t_{t'}$ (for $t'\in T'$, $t\in T$, $\mathrm{rk}'(t')=\mathrm{rk}(t)=\gamma$) such that $\bar q'\leq\bar q''$ and $q''_{t'}\Vdash_{\mathbb{P}_{\xi^*}}\underset{\sim}{f}{}^{t'}_t=g^t_{t'}$. Now, for $t\in T$ put

- $q_t=q''_t$ if $\mathrm{rk}(t)\leq\xi^*$, and
- $q_t=q''_{t\restriction\xi^*}{}^\frown g_t^{t\restriction\xi^*}$ if $\mathrm{rk}(t)=\gamma$.

It should be clear that $\bar q=\langle q_t:t\in T\rangle$ is a tree of conditions in $\bar{\mathbb{Q}}$, $\bar p\leq\bar q$ and for every $t\in T$ with $\mathrm{rk}(t)=\gamma$ we have $q_t\Vdash_{\mathbb{P}_\gamma}\underset{\sim}{A}\cap Z^r_\alpha\in d^r_\alpha$. This shows that X^* is included in the set X defined in the assumption (d), and hence $\bigcup\big\{Z^r_\alpha:\alpha\in X\big\}\in\mathrm{fil}(G^*)$. $\square$

Let $\underset{\sim}{A}$ be a $\mathbb{P}_\gamma$–name for a subset of λ such that $\Vdash_{\mathbb{P}_\gamma}\underset{\sim}{A}\in\big(\mathrm{fil}(G^*)\big)^+$ and let $p\in\mathbb{P}_\gamma$. We will find a condition $p^*\geq p$ such that $p^*\Vdash_{\mathbb{P}_\gamma}\underset{\sim}{A}\in\mathrm{fil}(G^*)$. It will be provided by the winning criterion $(\circledast)^{\rm tree}_{\mathbf{A}}$ of the game $\Game^{\rm tree\mathbf{A}}_{\bar\mu}(p,\bar{\mathbb{Q}})$ (see Definition 2.7; remember $\mathbb{P}_\gamma$ is reasonably* $A(\bar{\mathbb{Q}})$–bounding over $\bar\mu$ by Theorem 2.8).

Let $\mathbf{st}$ be a winning strategy of Generic in $\Game^{\rm tree\mathbf{A}}_{\bar\mu}(p,\bar{\mathbb{Q}})$, and for $\varepsilon\leq\gamma$ and $q\in\mathbb{P}_\varepsilon$ let us fix a winning strategy $\mathbf{st}(\varepsilon,q)$ of Complete in $\Game^\lambda_0(\mathbb{P}_\varepsilon,q)$ so that the coherence demands (i)–(iii) of Proposition 2.3 are satisfied.

We are going to describe a strategy $\mathbf{st}^{\boxplus}$ of INC in the game $\Game^{\boxplus}_{\bar\mu}(G^*)$. In the course of a play of $\Game^{\boxplus}_{\bar\mu}(G^*)$, INC will construct aside a play of $\Game^{\rm tree\mathbf{A}}_{\bar\mu}(p,\bar{\mathbb{Q}})$ in which

Generic plays according to $\mathbf{st}$. So suppose that INC and COM arrived to a stage $\alpha<\lambda$ of a play of $\Game^{\boxplus}_{\bar{\mu}}(G^*)$, and they have constructed

$(\circledast)^\alpha_1$ $\quad \big\langle I_\gamma, i_\gamma, \bar{u}_\gamma, \langle r_{\gamma,i}, r'_{\gamma,i}, (\beta_{\gamma,i}, Z_{\gamma,i}, d_{\gamma,i}) : i<i_\gamma\rangle : \gamma<\alpha\big\rangle.$

Also let us assume that INC (playing according to $\mathbf{st}^{\boxplus}$) has written aside a partial play

$(\circledast)^\alpha_2$ $\quad \big\langle \mathcal{T}_\gamma, \bar{p}^\gamma, \bar{q}^\gamma : \gamma<\alpha\big\rangle$

of $\Game^{\mathrm{tree}\mathbf{A}}_{\bar{\mu}}(p,\bar{\mathbb{Q}})$ (in which Generic plays according to $\mathbf{st}$). Let a standard tree $\mathcal{T}_\alpha$ and a tree of conditions $\bar{p}^\alpha=\langle p^\alpha_t : t\in T_\alpha\rangle$ be given to Generic by the strategy $\mathbf{st}$ in answer to $(\circledast)^\alpha_2$ (so $|T_\alpha|<\mu_\alpha$).

On the board of $\Game^{\boxplus}_{\bar{\mu}}(G^*)$, the strategy $\mathbf{st}^{\boxplus}$ instructs INC to play $I_\alpha \stackrel{\rm def}{=} \{t\in T_\alpha : \mathrm{rk}_\alpha(t)=\gamma\}$ and the $<^*_\chi$–first enumeration $\bar{u}_\alpha=\langle u_{\alpha,i} : i<i_\alpha\rangle$ of $[I_\alpha]^{<\omega}$ (so $i_\alpha<\mu_\alpha$). Now the two players start playing a subgame of length i_α to determine a sequence $\langle r_{\alpha,i}, r'_{\alpha,i}, (\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : i<i_\alpha\rangle$. During the subgame INC will construct aside a sequence $\langle \bar{q}^0_i, \bar{q}^1_i : i<i_\alpha\rangle$ of trees of conditions in $\mathbb{P}_\gamma$ so that

$(\circledast)_3$ $\bar{q}^\ell_i=\langle q^\ell_{t,i} : t\in T_\alpha\rangle$ (for $\ell<2$, $i<i_\alpha$) and for each $t\in T_\alpha$, the sequence $\langle q^0_{t,i}, q^1_{t,i} : i<i_\alpha\rangle$ is a legal play of $\Game^\lambda_0(\mathbb{P}_{\mathrm{rk}_\alpha(t)}, p^\alpha_t)$ in which Complete uses her winning strategy $\mathbf{st}(\mathrm{rk}_\alpha(t), p^\alpha_t)$.

Suppose that COM and INC arrive at level $i<i_\alpha$ of the subgame (of $\Game^{\boxplus}_{\bar{\mu}}(G^*)$) and

$(\circledast)^i_4$ $\quad \langle r_{\alpha,j}, r'_{\alpha,j}, (\beta_{\alpha,j}, Z_{\alpha,j}, d_{\alpha,j}) : j<i\rangle$ and $\langle \bar{q}^0_j, \bar{q}^1_j : j<i\rangle$

have been determined and COM has chosen $r_{\alpha,i}\in G^*$. INC's answer is given by $\mathbf{st}^{\boxplus}$ as follows. First, INC takes the $<^*_\chi$–first tree of conditions $\bar{q}^\diamond$ in $\bar{\mathbb{Q}}$ such that

$(\circledast)^{\rm a}_5$ $\bar{q}^\diamond=\langle q^\diamond_t : t\in T_\alpha\rangle$ and $q^\diamond_t\in\mathbb{P}_{\mathrm{rk}(t)}$ is an upper bound to $\{p^\alpha_t\}\cup\{q^1_{t,j} : j<i\}$ (for each $t\in T_\alpha$)

(remember $(\circledast)_3$). Then INC lets $X\subseteq C^{r_{\alpha,i}}$ to be the set of all $\beta\in C^{r_{\alpha,i}}$ greater than $\sup\Big(\bigcup\limits_{\gamma<\alpha}\bigcup\limits_{j<i_\gamma} Z_{\gamma,j}\cup\bigcup\limits_{j<i} Z_{\alpha,j}\Big)+890$ and such that

$(\circledast)^{\rm b}_5$ there is a tree of conditions $\bar{q}'$ in $\bar{\mathbb{Q}}$ such that $q^\diamond\leq\bar{q}'$ and

$$\text{if } t\in u_{\alpha,i}, \text{ then } q'_t\Vdash_{\mathbb{P}_\gamma} \underset{\sim}{A}\cap Z^{r_{\alpha,i}}_\beta\in d^{r_{\alpha,i}}_\beta.$$

Since $u_{\alpha,i}$ is finite, it follows from 3.2.1 that $\bigcup\big\{Z^{r_{\alpha,i}}_\beta : \beta\in X\big\}\in \mathrm{fil}(G^*)$. Then INC picks also the club C of λ such that $C\subseteq C^{r_{\alpha,i}}$ and $r_{\alpha,i}$ is restrictable to $\langle X,C\rangle$ (see Definition 1.7) and $\min(C)=\min(X)$, and his inning at the stage i of the subgame of $\Game^{\boxplus}_{\bar{\mu}}(G^*)$ is $r'_{\alpha,i}=r_{\alpha,i}{\restriction}\langle X,C\rangle$ (again, see Definition 1.7; note that $r'_{\alpha,i}\in G^*$ by 1.8).

After this COM answers with $(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i})\in\#\big(r'_{\alpha,i}\big)$, and then INC chooses (for the construction aside) the $<^*_\chi$–first tree of conditions $\bar{q}^0_i$ in $\bar{\mathbb{Q}}$ such that $\bar{q}^\diamond\leq\bar{q}^0_i$ and

$(\circledast)_6$ if $t\in u_{\alpha,i}$, then $q^0_{t,i}\Vdash_{\mathbb{P}_\gamma} \underset{\sim}{A}\cap Z^{r_{\alpha,i}}_{\beta_{\alpha,i}}\in d^{r_{\alpha,i}}_{\beta_{\alpha,i}}$.

Then $\bar{q}^1_i=\langle q^1_{t,i} : t\in T_\alpha\rangle$ is a tree of conditions determined by the demand in $(\circledast)_3$ and the strategies $\mathbf{st}(\mathrm{rk}_\alpha(t), p^\alpha_t)$ (for $t\in T_\alpha$); remember the coherence conditions of 2.3.

This completes the description of how INC plays in the subgame of stage α. After the subgame is finished, INC determines the move $\bar{q}^\alpha$ of Antigeneric in the play of $\Game^{\mathrm{tree}\mathbf{A}}_{\bar{\mu}}(p,\bar{\mathbb{Q}})$ he is constructing aside:

$(\circledast)_7$ $\bar{q}^\alpha$ is the $<^*_\chi$–first tree of conditions $\langle q^\alpha_t : t \in T_\alpha\rangle$ such that $\bar{q}^0_i \leq \bar{q}^1_i \leq \bar{q}^\alpha$ for all $i < i_\alpha$.

(There is such tree of conditions by $(\circledast)_3$; remember $i_\alpha < \mu_\alpha \leq \lambda$.)

This completes the description of the strategy $\mathbf{st}^{\boxplus}$. Since G^* is $\bar{\mu}$–super reasonable, $\mathbf{st}^{\boxplus}$ cannot be a winning strategy, so there is a play

$(\circledast)_8$ $\quad \big\langle I_\alpha, i_\alpha, \bar{u}_\alpha, \langle r_{\alpha,i}, r'_{\alpha,i}, (\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : i < i_\alpha\rangle : \alpha < \lambda\big\rangle$

of $\Game^{\boxplus}_{\bar{\mu}}(G^*)$ in which INC follows $\mathbf{st}^{\boxplus}$, but

$(\circledast)_9$ $\quad$ for some $r \in G^*$, for every $\langle j_\alpha : \alpha < \lambda\rangle \in \prod\limits_{\alpha<\lambda} I_\alpha$ we have

$$\big\{(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : \alpha < \lambda \ \&\ i < i_\alpha \ \&\ j_\alpha \in u_{\alpha,i}\big\} \leq^* \#(r).$$

Let $\langle \mathcal{T}_\alpha, \bar{p}^\alpha, \bar{q}^\alpha : \alpha < \lambda\rangle$ be the play of $\Game^{\mathrm{tree}\mathbf{A}}_{\bar{\mu}}(p, \bar{\mathbb{Q}})$ constructed aside by INC (so this is a play in which Generic uses her winning strategy $\mathbf{st}$). Since Generic won that play, there is a condition $p^* \in \mathbb{P}_\gamma$ stronger than p and such that for each $\alpha < \lambda$ the set $\{q^\alpha_t : t \in T_\alpha \ \&\ \mathrm{rk}_\alpha(t) = \gamma\}$ is pre-dense above p^*. Note that if we show that

$(\circledast)_{10}$ $\quad \Vdash_{\mathbb{P}_\gamma} \big(\forall \bar{j} \in \prod\limits_{\alpha<\lambda} I_\alpha\big)\big(\big\{(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : \alpha < \lambda \ \&\ i < i_\alpha \ \&\ j_\alpha \in u_{\alpha,i}\big\} \leq^* r\big)$

then we will be able to conclude that $p^* \Vdash \underset{\sim}{A} \in \mathrm{fil}(r)$ (remember $(\circledast)_6 + (\circledast)_7$ and 1.10), finishing the proof of the Theorem. So let us argue that $(\circledast)_{10}$ holds true.

It follows from the description of $\mathbf{st}^{\boxplus}$ (see the description of X after $(\circledast)^{\mathrm{a}}_5$) that we may choose a continuous increasing sequence $\langle \delta_\alpha : \alpha < \lambda\rangle \subseteq \lambda$ such that

$$\big(\forall \alpha < \lambda\big)\big(\delta_\alpha \leq \beta_{\alpha,0} \leq \sup\big(\bigcup_{i<i_\alpha} Z_{\alpha,i}\big) < \delta_{\alpha+1}\big).$$

Now, we will say that $\beta \in C^r$ *is a sick case* whenever there are $\alpha_0 < \alpha_1 < \lambda$ and $B \in d^r_\beta$ such that $Z^r_\beta \subseteq [\delta_{\alpha_0}, \delta_{\alpha_1})$ and

$$\big(\forall \alpha \in [\alpha_0, \alpha_1)\big)\big(\exists t \in I_\alpha\big)\big(\forall i < i_\alpha\big)\big(t \notin u_{\alpha,i} \ \text{ or } \ B \cap Z_{\alpha,i} \notin d_{\alpha,i}\big).$$

Using 1.10(2) one can easily verify that the following two conditions are equivalent:

$(\circledast)^{\mathrm{one}}_{11}$ there is $\langle j_\alpha : \alpha < \lambda\rangle \in \prod\limits_{\alpha<\lambda} I_\alpha$ such that

$$\big\{(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : \alpha < \lambda \ \&\ i < i_\alpha \ \&\ j_\alpha \in u_{\alpha,i}\big\} \not\leq^* r,$$

$(\circledast)^{\mathrm{two}}_{11}$ there are λ many sick cases of $\beta \in C^r$.

Since the forcing with $\mathbb{P}_\gamma$ does not add bounded subsets of λ, being a sick case is absolute between $\mathbf{V}$ and $\mathbf{V}^{\mathbb{P}_\gamma}$. So we may conclude (from $(\circledast)_9$) that $(\circledast)_{10}$ holds true and thus the proof of Theorem 3.2 is complete. $\square$

Theorem 3.3. *Assume (i) and (ii) of 3.2 and*

(α) $\bar{\kappa} = \langle \kappa_\alpha : \alpha < \lambda\rangle$ is a sequence of regular cardinals such that for each $\alpha < \lambda$:

$$\mu_\alpha \leq \kappa_\alpha \leq \lambda \quad \text{and} \quad (\forall \mu < \mu_\alpha)(2^\mu < \kappa_\alpha),$$

(β) $G^* \subseteq \mathbb{Q}^0_\lambda$ is $\bar{\kappa}$–super reasonable.

Then $\Vdash_{\mathbb{P}_\gamma}$ " G^* is $\bar{\mu}$–strongly reasonable ".

Proof. First of all note that the forcing notion $\mathbb{P}_\gamma$ is reasonably* $A(\bar{\mathbb{Q}})$–bounding over $\bar{\mu}$ and λ–proper (see 2.8). Therefore $\Vdash_{\mathbb{P}_\gamma}$ " $\big([G^*]^{\leq\lambda}\big)^{\mathbf{V}}$ is cofinal in $[G^*]^{\leq\lambda}$", and consequently $\Vdash_{\mathbb{P}_\gamma}$ " G^* is $(<\lambda^+)$–directed (with respect to $\leq^0$)".

Suppose that $\underset{\sim}{\mathbf{st}}^\oplus$ is a $\mathbb{P}_\gamma$–name, $p \in \mathbb{P}_\gamma$ and

$$\Vdash_{\mathbb{P}_\gamma} \text{“ } \underset{\sim}{\mathbf{st}}^\oplus \text{ is a strategy of INC in } \partial^\oplus_{\bar{\mu}}(G^*) \text{ such that all values given by it are from } \mathbf{V} \text{ ”.}$$

We are going to find a condition $p^* \geq p$ and a $\mathbb{P}_\gamma$–name $\underset{\sim}{g}_\lambda$ such that

$p^* \Vdash_{\mathbb{P}_\gamma}$ “ $\underset{\sim}{g}_\lambda$ is a play of $\partial^\oplus_{\bar{\mu}}(G^*)$ in which INC uses $\underset{\sim}{\mathbf{st}}^\oplus$ but COM wins the play ”.

The condition p^* will be provided by the winning criterion $(\circledast)^{\rm tree}_{\mathbf{A}}$ of the game $\partial^{\rm tree\mathbf{A}}_{\bar{\mu}}(p, \bar{\mathbb{Q}})$ (see Definition 2.7).

In the rest of the proof whenever we say “INC chooses/picks x such that” we mean “INC chooses/picks the $<^*_\chi$–first x such that”. Let us fix

(i) a winning strategy $\mathbf{st}$ of Generic in $\partial^{\rm tree\mathbf{A}}_{\bar{\mu}}(p, \bar{\mathbb{Q}})$,
(ii) winning strategies $\mathbf{st}(\varepsilon, q)$ of Complete in $\partial^\lambda_0(\mathbb{P}_\varepsilon, q)$ (for $\varepsilon \leq \gamma$, $q \in \mathbb{P}_\varepsilon$) such that the coherence conditions of 2.3 are satisfied.

We are going to describe a strategy $\mathbf{st}^\boxplus$ of INC in the game $\partial^\boxplus_{\bar{\kappa}}(G^*)$. In the course of a play of $\partial^\boxplus_{\bar{\kappa}}(G^*)$, INC will simulate a play of $\partial^{\rm tree\mathbf{A}}_{\bar{\mu}}(p, \bar{\mathbb{Q}})$ and he will consider names for partial plays of $\partial^\oplus_{\bar{\mu}}(G^*)$ in which INC uses $\underset{\sim}{\mathbf{st}}^\oplus$. Thus players INC/COM will appear in the play of $\partial^\boxplus_{\bar{\kappa}}(G^*)$ in $\mathbf{V}$ and in the play of $\partial^\oplus_{\bar{\mu}}(G^*)$ in $\mathbf{V}^{\mathbb{P}_\gamma}$. To avoid confusion we will refer to them as $\text{COM}^{\mathbf{V}}, \text{INC}^{\mathbf{V}}$ for $\partial^\boxplus_{\bar{\kappa}}(G^*)$ (in $\mathbf{V}$) and $\text{COM}^{\mathbf{V}^{\mathbb{P}_\gamma}}, \text{INC}^{\mathbf{V}^{\mathbb{P}_\gamma}}$ for $\partial^\oplus_{\bar{\mu}}(G^*)$ (in $\mathbf{V}^{\mathbb{P}_\gamma}$).

So suppose that $\text{INC}^{\mathbf{V}}$ and $\text{COM}^{\mathbf{V}}$ arrived at a stage $\alpha < \lambda$ of the play of $\partial^\boxplus_{\bar{\kappa}}(G^*)$ (in $\mathbf{V}$), and $\text{INC}^{\mathbf{V}}$ (playing according to $\mathbf{st}^\boxplus$) has written aside:

$(\oplus)^\alpha_1$ a partial play $\langle \mathcal{T}_\beta, \bar{p}^\beta, \bar{q}^\beta : \beta < \alpha\rangle$ of $\partial^{\rm tree\mathbf{A}}_{\bar{\mu}}(p, \bar{\mathbb{Q}})$ in which Generic plays according to $\mathbf{st}$, and
$(\oplus)^\alpha_2$ a $\mathbb{P}_\gamma$–name $\underset{\sim}{g}_\alpha = \langle \underset{\sim}{I}_\beta, \underset{\sim}{i}_\beta, \underset{\sim}{\bar{u}}_\beta, \underset{\sim}{x}_\beta : \beta < \alpha\rangle$ of a partial play of $\partial^\oplus_{\bar{\mu}}(G^*)$ (in $\mathbf{V}^{\mathbb{P}_\gamma}$) in which $\text{INC}^{\mathbf{V}^{\mathbb{P}_\gamma}}$ uses the strategy $\underset{\sim}{\mathbf{st}}^\oplus$,
$(\oplus)^\alpha_3$ ordinals $i_\beta < \mu_\beta$ such that $q^\beta_t \Vdash \underset{\sim}{i}_\beta = i_\beta$ for every $t \in T_\beta$ with $\text{rk}_\beta(t) = \gamma$ (for $\beta < \alpha$).

Note that $\underset{\sim}{I}_\beta$ is a $\mathbb{P}_\gamma$–name for a set of size $< \mu_\beta$ from $\mathbf{V}$, $\underset{\sim}{\bar{u}}_\beta$ is a $\mathbb{P}_\gamma$–name for an $\underset{\sim}{i}_\beta$–sequence of finite subsets of $\underset{\sim}{I}_\beta$ and $\underset{\sim}{x}_\beta$ is a $\mathbb{P}_\gamma$–name for the result of the subgame of length $\underset{\sim}{i}_\beta$ of level β.

Let $\underset{\sim}{I}_\alpha$ be a $\mathbb{P}_\gamma$–name for the answer by $\underset{\sim}{\mathbf{st}}^\oplus$ to the play $\underset{\sim}{g}_\alpha$ of $\partial^\oplus_{\bar{\mu}}(G^*)$ (in $\mathbf{V}^{\mathbb{P}_\gamma}$).

Let $\mathcal{T}_\alpha$ and $\bar{p}^\alpha = \langle p^\alpha_t : t \in T_\alpha\rangle$ be given to Generic by the strategy $\mathbf{st}$ as an answer to $(\oplus)^\alpha_1$. Let $\bar{q}^\diamond = \langle q^\diamond_t : t \in T_\alpha\rangle$ be a tree of conditions in $\bar{\mathbb{Q}}$ such that

$(\oplus)^{\rm a}_4$ $\bar{p}^\alpha \leq \bar{q}^\diamond$ and $q^\diamond_{t_0}, q^\diamond_{t_1}$ are incompatible whenever $t_0, t_1 \in T_\alpha$, $\text{rk}_\alpha(t_0) = \text{rk}_\alpha(t_1)$ but $t_0 \neq t_1$,
$(\oplus)^{\rm b}_4$ for every $t \in T_\alpha$ with $\text{rk}_\alpha(t) = \gamma$ the condition $q^\diamond_t$ decides the value of $\underset{\sim}{I}_\alpha$, say $q^\diamond_t \Vdash_{\mathbb{P}_\gamma}$ “ $\underset{\sim}{I}_\alpha = I^t_\alpha$ ”.

(Note that $\Vdash_{\mathbb{P}_\gamma} \underset{\sim}{I}_\alpha \in \mathbf{V}$ by the choice of $\underset{\sim}{\mathbf{st}}^\oplus$; remember 2.5.)

In the play of $\partial^\boxplus_{\bar{\kappa}}(G^*)$, the strategy $\mathbf{st}^\boxplus$ instructs $\text{INC}^{\mathbf{V}}$ to choose $I_\alpha = \prod\{I^t_\alpha : t \in T_\alpha \ \&\ \text{rk}_\alpha(t) = \gamma\}$ and an enumeration $\bar{u}_\alpha = \langle u_{\alpha,i} : i < i_\alpha\rangle$ of $[I_\alpha]^{<\omega}$. Note that $|I^t_\alpha| < \mu_\alpha$ for all relevant $t \in T_\alpha$ and $|T_\alpha| < \mu_\alpha$, so by our assumptions on μ_α and κ_α we know that $|I_\alpha| < \kappa_\alpha$ (so also $i_\alpha < \kappa_\alpha$).

Then, in the play of $\eth^{\oplus}_{\bar{\mu}}(G^*)$, $\mathrm{INC}^{\mathbf{V}^{\mathbb{P}_\gamma}}$ pretends that $\mathrm{COM}^{\mathbf{V}^{\mathbb{P}_\gamma}}$ played an ordinal $\underset{\sim}{i}_\alpha \in [i_\alpha, \lambda)$ and $\bar{\underset{\sim}{u}}_\alpha = \langle \underset{\sim}{u}_{\alpha,i} : i < \underset{\sim}{i}_\alpha\rangle$ such that

$$\Vdash_{\mathbb{P}_\gamma} \text{“ } \bar{\underset{\sim}{u}}_\alpha \subseteq [\underset{\sim}{I}_\alpha]^{<\omega} \ \text{ and } \ \bigcup\{\underset{\sim}{u}_{\alpha,i} : i < \underset{\sim}{i}_\alpha\} = \underset{\sim}{I}_\alpha \text{ ”}$$

and for each $t \in T_\alpha$ with $\mathrm{rk}_\alpha(t) = \gamma$ we have

$$q^\diamond_t \Vdash_{\mathbb{P}_\gamma} \text{“ } \underset{\sim}{i}_\alpha = i_\alpha \text{ and } \underset{\sim}{u}_{\alpha,i} = \{c(t) : c \in u_{\alpha,i}\} \text{ for } i < i_\alpha \text{ ”}.$$

Now, both in $\eth^{\oplus}_{\bar{\mu}}(G^*)$ of $\mathbf{V}^{\mathbb{P}_\gamma}$ and in $\eth^{\boxplus}_{\bar{\kappa}}(G^*)$ of $\mathbf{V}$ the two players start a subgame. The length of the subgame in $\mathbf{V}^{\mathbb{P}_\gamma}$ may be longer than i_α, but we will restrict our attention to the first i_α steps of that subgame. In our *active case* we will have $\underset{\sim}{i}_\alpha = i_\alpha$, see the choice of $\underset{\sim}{i}_\alpha$ above. When playing the subgame, $\mathrm{INC}^{\mathbf{V}}$ will build a sequence $\langle \bar{q}^0_i, \bar{q}^1_i : i < i_\alpha\rangle$ of trees of conditions in $\bar{\mathbb{Q}}$ such that (in addition to demands stated later):

- $(\oplus)^{\mathrm{a}}_5$ $\bar{q}^\ell_j = \langle q^\ell_{t,j} : t \in T_\alpha\rangle$, $\bar{q}^\diamond \le \bar{q}^0_j \le \bar{q}^1_j \le \bar{q}^0_i$ for $\ell < 2$, $j < i < i_\alpha$, and
- $(\oplus)^{\mathrm{b}}_5$ for each $t \in T_\alpha$, the sequence $\langle q^0_{t,i}, q^1_{t,i} : i < i_\alpha\rangle$ is a legal play of $\eth^\lambda_0(\mathbb{P}_{\mathrm{rk}_\alpha(t)}, q^\diamond_t)$ in which Complete uses her winning strategy $\mathbf{st}(\mathrm{rk}_\alpha(t), q^\diamond_t)$.

He (as $\mathrm{INC}^{\mathbf{V}^{\mathbb{P}_\gamma}}$) will also construct a name for a play of a subgame of $\eth^{\oplus}_{\bar{\mu}}(G^*)$ of $\mathbf{V}^{\mathbb{P}_\gamma}$ for this stage.

Suppose that $\mathrm{INC}^{\mathbf{V}}$ and $\mathrm{COM}^{\mathbf{V}}$ have arrived to a stage $i < i_\alpha$ of the subgame and $\mathrm{INC}^{\mathbf{V}}$ has determined aside $\bar{q}^\ell_j$ for $j < i$, $\ell < 2$ and a $\mathbb{P}_\gamma$–name $\langle \underset{\sim}{z}^\alpha_j : j < i\rangle$ for a partial play of the subgame of $\eth^{\oplus}_{\bar{\mu}}(G^*)$ of $\mathbf{V}^{\mathbb{P}_\gamma}$. Now $\mathrm{COM}^{\mathbf{V}}$ chooses $r_{\alpha,i} \in G^*$ which $\mathrm{INC}^{\mathbf{V}}$ passes to $\mathrm{INC}^{\mathbf{V}^{\mathbb{P}_\gamma}}$ as an inning of $\mathrm{COM}^{\mathbf{V}^{\mathbb{P}_\gamma}}$ at the ith step of the subgame of level α of $\eth^{\oplus}_{\bar{\mu}}(G^*)$ in $\mathbf{V}^{\mathbb{P}_\gamma}$. There the strategy $\underset{\sim}{\mathbf{st}}^{\oplus}$ gives $\mathrm{INC}^{\mathbf{V}^{\mathbb{P}_\gamma}}$ an answer $\underset{\sim}{\delta}_{\alpha,i} < \lambda$.

Next, $\mathrm{INC}^{\mathbf{V}}$ picks a tree of conditions $\bar{q}^0_i = \langle q^0_{t,i} : t \in T_\alpha\rangle$ in $\bar{\mathbb{Q}}$ such that

- $(\oplus)^{\mathrm{a}}_6$ $(\forall j < i)(\bar{q}^1_j \le \bar{q}^0_i)$ and $\bar{q}^\diamond \le \bar{q}^0_i$, and
- $(\oplus)^{\mathrm{b}}_6$ for every $t \in T_\alpha$ with $\mathrm{rk}_\alpha(t) = \gamma$, the condition $q^0_{t,i}$ decides the value of $\underset{\sim}{\delta}_{\alpha,i}$, say $q^0_{t,i} \Vdash_{\mathbb{P}_\gamma} \underset{\sim}{\delta}_{\alpha,i} = \delta^t_{\alpha,i}$.

Then $\mathrm{INC}^{\mathbf{V}}$ lets

$$\delta^*_{\alpha,i} = \sup\Big(\big\{\delta^t_{\alpha,i} : t \in T_\alpha \ \&\ \mathrm{rk}_\alpha(t) = \gamma\big\} \cup \bigcup_{\beta<\alpha}\bigcup_{j<i_\beta} Z_{\beta,j} \cup \bigcup_{j<i} Z_{\alpha,j}\Big) + 890$$

and in the subgame of $\eth^{\boxplus}_{\bar{\kappa}}(G^*)$ (in $\mathbf{V}$) he is instructed to put $r'_{\alpha,i}$ such that

$$C^{r'_{\alpha,i}} = C^{r_{\alpha,i}} \setminus \delta^*_{\alpha,i} \quad \text{ and } \quad d^{r'_{\alpha,i}}_\beta = d^{r_{\alpha,i}}_\beta \text{ for } \beta \in C^{r'_{\alpha,i}}.$$

(Note that $r'_{\alpha,i} \in G^*$ by 1.11(2)(ii).)

After this $\mathrm{COM}^{\mathbf{V}}$ chooses $(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) \in \#(r'_{\alpha,i})$, so $\beta_{\alpha,i} \in C^{r_{\alpha,i}}$, $\beta_{\alpha,i} \ge \delta^*_{\alpha,i}$ and $d_{\alpha,i} = d^{r_{\alpha,i}}_{\beta_{\alpha,i}}$. Next $\mathrm{INC}^{\mathbf{V}}$ lets

- $\bar{q}^1_i$ be the tree of conditions in $\bar{\mathbb{Q}}$ fully determined by demand $(\oplus)^{\mathrm{b}}_5$ and
- $\underset{\sim}{z}^\alpha_i$ be a $\mathbb{P}_\gamma$–name for a legal result of stage i of the subgame of level α of $\eth^{\oplus}_{\bar{\mu}}(G^*)$ in $\mathbf{V}^{\mathbb{P}_\gamma}$ such that for each $t \in T_\alpha$ with $\mathrm{rk}_\alpha(t) = \gamma$ we have

$$q^0_{t,i} \Vdash_{\mathbb{P}_\gamma} \underset{\sim}{z}^\alpha_i = \big(r_{\alpha,i}, \underset{\sim}{\delta}_{\alpha,i}, (\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i})\big).$$

Then the subgame continues.

After i_α steps of the subgame, $\mathrm{INC}^{\mathbf{V}}$ chooses a tree of conditions $\bar{q}^\alpha = \langle q^\alpha_t : t \in T_\alpha\rangle$ in $\bar{\mathbb{Q}}$ such that $(\forall i < i_\alpha)(\bar{q}^1_i \leq \bar{q}^\alpha)$ and he also lets $\underaccent{\tilde}{x}_\alpha$ be a $\mathbb{P}_\gamma$–name for the result of the subgame of level α of $\partial^{\oplus}_{\bar{\mu}}(G^*)$ in $\mathbf{V}^{\mathbb{P}_\gamma}$ such that $\underaccent{\tilde}{x}_\alpha{\restriction} i_\alpha = \langle \underaccent{\tilde}{z}^\alpha_i : i < i_\alpha\rangle$. Note that all the objects described by $(\oplus)^{\alpha+1}_1$–$(\oplus)^{\alpha+1}_3$ are determined now.

This completes the description of the strategy $\mathbf{st}^{\boxplus}$ of INC (i.e., $\mathrm{INC}^{\mathbf{V}}$) in $\partial^{\boxplus}_{\bar{\kappa}}(G^*)$. Since G^* is $\bar{\mu}$–super reasonable, this strategy cannot be a winning one, so there is a play

$(\oplus)_7$ $\big\langle I_\alpha, i_\alpha, \bar{u}_\alpha, \langle r_{\alpha,i}, r'_{\alpha,i}, (\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : i < i_\alpha\rangle : \alpha < \lambda\big\rangle$

of $\partial^{\boxplus}_{\bar{\kappa}}(G^*)$ in which INC follows $\mathbf{st}^{\boxplus}$, but

$(\oplus)_8$ for some $r \in G^*$, for every $\langle j_\alpha : \alpha < \lambda\rangle \in \prod\limits_{\alpha<\lambda} I_\alpha$ we have

$$\{(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : \alpha < \lambda \ \&\ i < i_\alpha \ \&\ j_\alpha \in u_{\alpha,i}\} \leq^* \#(r).$$

Exactly as in the proof of Theorem 3.2 we may argue that

$(\oplus)_9$ $\Vdash_{\mathbb{P}_\gamma} \big(\forall \bar{j} \in \prod\limits_{\alpha<\lambda} I_\alpha\big)\big(\big\{(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) \colon \alpha < \lambda \ \&\ i < i_\alpha \ \&\ j_\alpha \in u_{\alpha,i}\big\} \leq^* r\big)$.

(See $(\circledast)_{10}$ in the proof of 3.2.)

Let $\langle \mathcal{T}_\alpha, \bar{p}^\alpha, \bar{q}^\alpha : \alpha < \lambda\rangle$ be the play of $\partial^{\mathrm{tree}\mathbf{A}}_{\bar{\mu}}(p, \bar{\mathbb{Q}})$ constructed aside by INC. Generic won that play, so there is a condition $p^* \in \mathbb{P}_\gamma$ stronger than p and such that for each $\alpha < \lambda$ the set $\{q^\alpha_t : t \in T_\alpha \ \&\ \mathrm{rk}_\alpha(t) = \gamma\}$ is pre-dense above p^*. Also, let $\underaccent{\tilde}{g}_\lambda$ be the $\mathbb{P}_\gamma$–name of a play of $\partial^{\oplus}_{\bar{\mu}}(G^*)$ (in $\mathbf{V}^{\mathbb{P}_\gamma}$) constructed aside in the same run of $\partial^{\boxplus}_{\bar{\kappa}}(G^*)$ (see $(\oplus)_2$). We are going to argue that

$(\oplus)_{10}$ the condition p^* forces (in $\mathbb{P}_\gamma$) that

$$\big(\forall \bar{j} \in \prod_{\alpha<\lambda} \underaccent{\tilde}{I}_\alpha\big)\big(\{(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : \alpha < \lambda \ \&\ i < \underaccent{\tilde}{i}_\alpha \ \&\ j_\alpha \in \underaccent{\tilde}{u}_{\alpha,i}\} \leq^* \#(r)\big),$$

that is

$$p^* \Vdash_{\mathbb{P}_\gamma} \text{“ } \mathrm{COM}^{\mathbf{V}^{\mathbb{P}_\gamma}} \text{ wins the play } \underaccent{\tilde}{g}_\lambda \text{ as witnessed by } r \text{ ”}.$$

So suppose that $G \subseteq \mathbb{P}_\gamma$ is generic over $\mathbf{V}$, $p^* \in G$, and let us work in $\mathbf{V}[G]$. For every $\alpha < \lambda$ there is a unique $t = t(\alpha) \in T_\alpha$ such that $\mathrm{rk}_\alpha(t) = \gamma$ and $q^\alpha_t \in G$, and thus $\big(\underaccent{\tilde}{I}_\alpha\big)^G = I^t_\alpha$, $\big(\underaccent{\tilde}{i}_\alpha\big)^G = i_\alpha$ and $\big(\bar{\underaccent{\tilde}{u}}_\alpha\big)^G = \langle (\underaccent{\tilde}{u}_{\alpha,i})^G : i < i_\alpha\rangle$, where $\big(\underaccent{\tilde}{u}_{\alpha,i}\big)^G = \{c(t) : c \in u_{\alpha,i}\} \subseteq I^t_\alpha$. Suppose that $\bar{j} = \langle j_\alpha : \alpha < \lambda\rangle \in \prod\limits_{\alpha<\lambda} I^{t(\alpha)}_\alpha$. For each $\alpha < \lambda$ fix $j^*_\alpha \in I_\alpha = \prod\{I^t_\alpha : t \in T_\alpha \ \&\ \mathrm{rk}_\alpha(t) = \gamma\}$ such that $j^*_\alpha(t(\alpha)) = j_\alpha$. Note that if $j^*_\alpha \in u_{\alpha,i}$, $i < i_\alpha$, then $j_\alpha \in \big(\underaccent{\tilde}{u}_{\alpha,i}\big)^G$ and therefore

$$\begin{array}{rl}\{(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : \alpha < \lambda \ \&\ i < i_\alpha \ \&\ j_\alpha \in \big(\underaccent{\tilde}{u}_{\alpha,i}\big)^G\} & \leq^* \\ \{(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : \alpha < \lambda \ \&\ i < i_\alpha \ \&\ j^*_\alpha \in u_{\alpha,i}\} & \leq^* \quad r\end{array}$$

(remember $(\oplus)_9$). Now $(\oplus)_{10}$ follows and the proof of the theorem is complete. $\square$

Corollary 3.4. *Assume that λ is a strongly inaccessible cardinal. Then there is a forcing notion $\mathbb{P}$ such that*

$\Vdash_{\mathbb{P}}$ “ *λ is strongly inaccessible and $2^\lambda = \lambda^{++}$ and*
there is a strongly reasonable family $G^* \subseteq \mathbb{Q}^0_\lambda$ such that
$\mathrm{fil}(G^*)$ is an ultrafilter on λ and $|G^*| = \lambda^+$, in particular there is
a very reasonable ultrafilter on λ with generating system of size $< 2^\lambda$ ”

Proof. We may start with a universe $\mathbf{V}$ in which $\diamondsuit_{S^{\lambda^+}_\lambda}$ holds (and λ is strongly inaccessible). It follows from 1.15 that (in $\mathbf{V}$) there is a $\leq^*$–increasing sequence $\langle r_\alpha : \alpha < \lambda^+\rangle \subseteq \mathbb{Q}^0_\lambda$ such that $G^* \stackrel{\rm def}{=} \{r \in \mathbb{Q}^0_\lambda : (\exists \alpha < \lambda^+)(r \leq^* r_\alpha)\}$ is super reasonable and $\mathrm{fil}(G^{**})$ is an ultrafilter on λ.

Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \lambda^{++}\rangle$ be a λ–support iteration of the forcing notion $\mathbb{Q}^{\rm tree}_{\mathcal{D}_\lambda}(K_1, \Sigma_1)$ defined in the proof of [7, Prop. B.8.5]. This forcing is reasonably A–bounding (by [6, Prop. 3.2] and [7, Thm B.6.5]), so we may use Theorems 3.2 and 3.3 to conclude that

$$\Vdash_{\mathbb{P}_{\lambda^{++}}} \text{“ } G^* \text{ is strongly reasonable, } |G^*| = \lambda^+ < 2^\lambda \text{ and } \mathrm{fil}(G^*) \text{ is ultrafilter on } \lambda \text{ ”}.$$

If one analyzes the proof of Theorem 3.3, one may notice that even

$$\Vdash_{\mathbb{P}_{\lambda^{++}}} \text{“ } \{r_\alpha : \alpha < \lambda^+\} \text{ is strongly reasonable ”}.$$

□

4. A feature, not a bug

One may wonder if Theorems 3.2, 3.3 could be improved by replacing the assumption that we are working with the iteration of reasonably A–bounding forcings by, say, just dealing with a nicely double **a**–bounding forcing. A result of that sort would be more natural and the fact that we had to refer to an iteration-specific property could be seen as some lack of knowledge. However, this is *a feature, not a bug* as nicely double **a**–bounding forcing notions may cause that $\mathrm{fil}(G^*)$ is not an ultrafilter anymore.

In this section we assume that λ is a strongly inaccessible cardinal.

Definition 4.1. (1) Let $\mathbb{P}^*$ consist of all pairs $p = (\eta^p, C^p)$ such that $\eta^p : \lambda \longrightarrow \{-1, 1\}$ and C is a club of λ. A binary relation $\leq = \leq_{\mathbb{P}^*}$ on $\mathbb{P}^*$ is defined by letting $p \leq q$ if and only if

(α) $C^q \subseteq C^p$, $\eta^q{\restriction}\min(C^p) = \eta{\restriction}\min(C^q)$, and

(β) for every successive members $\alpha < \beta$ of C^p we have

$$\big(\forall \gamma \in [\alpha, \beta)\big)\big(\eta^q(\gamma) = \frac{\eta^p(\alpha)}{\eta^q(\alpha)} \cdot \eta^p(\gamma)\big).$$

(2) For $p \in \mathbb{P}^*$ and $\alpha \in C^p$ let

$$\mathrm{pos}(p, \alpha) \stackrel{\rm def}{=} \big\{\eta^q{\restriction}\alpha : q \in \mathbb{P}^* \ \&\ p \leq q\big\}.$$

(3) For $p \in \mathbb{P}^*$, $\alpha < \lambda$ and $\nu : \alpha \longrightarrow \{-1, 1\}$ let $\nu *_\alpha p = (\nu{}^\frown\eta^p{\restriction}[\alpha, \lambda), C^p \setminus \alpha)$. (Plainly, $\nu *_\alpha p \in \mathbb{P}^*$.)

Remark 4.2. $\mathbb{P}^*$ is a natural generalization of the forcing notion used by Goldstern and Shelah [4] to the context of uncountable cardinals.

Proposition 4.3. *Let $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda\rangle$, $\mu_\alpha = 2^{|\alpha| + \aleph_0}$ (for $\alpha < \lambda$). Then $\mathbb{P}^*$ is a nicely double* **a***–bounding over $\bar{\mu}$ forcing notion. Also $|\mathbb{P}^*| = 2^\lambda$.*

Proof. One easily verifies that the relation $\leq_{\mathbb{P}^*}$ is transitive and reflexive, also plainly $|\mathbb{P}^*| = 2^\lambda$.

Claim 4.3.1. $\mathbb{P}^*$ *is* $({<}\lambda)$*–complete.*

Proof of the Claim. Suppose that $\delta < \lambda$ and $\langle p_\xi : \xi < \delta\rangle$ is a $\leq_{\mathbb{P}^*}$–increasing sequence of conditions in $\mathbb{P}^*$. Let $C = \bigcap\limits_{\xi<\delta} C^{p_\xi}$ (it is a club of λ) and let $\eta : \lambda \longrightarrow \{-1,1\}$ be defined by

- if $\gamma < \min(C)$ and $\zeta = \min\big(\xi < \delta : \gamma < \min(C^{p_\xi})\big)$, then $\eta(\gamma) = \eta^{p_\zeta}(\gamma)$,
- if $\alpha < \beta$ are successive members of C, $\alpha \leq \gamma < \beta$ and $\zeta = \min\big(\xi < \delta : \gamma < \min\big(C^{p_\xi} \setminus (\alpha+1)\big)\big)$, then $\eta(\gamma) = \eta^{p_\zeta}(\alpha) \cdot \eta^{p_\zeta}(\gamma)$.

Plainly, η is well defined and $q \stackrel{\rm def}{=} (\eta, C) \in \mathbb{P}^*$. We claim that $(\forall \xi < \delta)(p_\xi \leq q)$. To this end suppose $\xi < \delta$. Clearly $C \subseteq C^{p_\xi}$. Now, if $\gamma < \min(C^{p_\xi})$, then $\eta(\gamma) = \eta^{p_\zeta}(\gamma)$ for some $\zeta \leq \xi$ such that $\gamma < \min(C^{p_\zeta})$. Since $p_\zeta \leq p_\xi$, we have $\eta^{p_\zeta}(\gamma) = \eta^{p_\xi}(\gamma)$ and thus $\eta^{p_\xi}(\gamma) = \eta(\gamma)$.

Next, suppose that $\alpha < \beta$ are successive members of C^{p_ξ} and $\alpha \leq \gamma < \beta$. If $\gamma < \min(C)$ and $\zeta = \min\big(\varepsilon < \delta : \gamma < \min(C^{p_\varepsilon})\big)$, then $\zeta > \xi$, $\eta(\alpha) = \eta^{p_\zeta}(\alpha)$ and

$(*)^1$ $\eta(\gamma) = \eta^{p_\zeta}(\gamma) = \frac{\eta^{p_\xi}(\alpha)}{\eta^{p_\zeta}(\alpha)} \cdot \eta^{p_\xi}(\gamma) = \frac{\eta^{p_\xi}(\alpha)}{\eta(\alpha)} \cdot \eta^{p_\xi}(\gamma)$.

So assume $C \cap \beta \neq \emptyset$ and let $\alpha' < \beta'$ be successive members of C such that $\alpha' \leq \alpha \leq \gamma < \beta \leq \beta'$. Let $\zeta = \min\big(\varepsilon < \delta : \gamma < \min\big(C^{p_\varepsilon} \setminus (\alpha'+1)\big)\big)$. If $\alpha = \alpha'$, then $\zeta \leq \xi$ and

$(*)^2$ $\eta(\gamma) = \eta^{p_\zeta}(\alpha) \cdot \eta^{p_\zeta}(\gamma) = \eta^{p_\zeta}(\alpha) \cdot \frac{\eta^{p_\xi}(\alpha)}{\eta^{p_\zeta}(\alpha)} \cdot \eta^{p_\xi}(\gamma) = \eta^{p_\xi}(\alpha) \cdot \eta^{p_\xi}(\gamma) = \frac{\eta^{p_\xi}(\alpha)}{\eta(\alpha)} \cdot \eta^{p_\xi}(\gamma)$

(as $\eta(\alpha) = \eta(\alpha') = 1$). If $\alpha' < \alpha$, then $\xi < \zeta$ and $\eta(\alpha) = \eta^{p_\zeta}(\alpha') \cdot \eta^{p_\zeta}(\alpha)$, and hence

$(*)^3$ $\eta(\gamma) = \eta^{p_\zeta}(\alpha') \cdot \eta^{p_\zeta}(\gamma) = \frac{\eta(\alpha)}{\eta^{p_\zeta}(\alpha)} \cdot \eta^{p_\zeta}(\gamma) = \frac{\eta(\alpha)}{\eta^{p_\zeta}(\alpha)} \cdot \frac{\eta^{p_\zeta}(\alpha)}{\eta^{p_\xi}(\alpha)} \cdot \eta^{p_\xi}(\gamma) = \frac{\eta^{p_\xi}(\alpha)}{\eta(\alpha)} \cdot \eta^{p_\xi}(\gamma)$.

Clearly $(*)^1$–$(*)^3$ are what we need to justify 4.1(1β) and conclude $p_\xi \leq q$. $\square$

Claim 4.3.2. *Let $p \in \mathbb{P}^*$. Then Generic has a nice winning strategy in the game $\Game^{\rm rc\mathbf{2a}}_{\bar{\mu}}(p, \mathbb{P}^*)$ (see Definition 2.9).*

Proof of the Claim. We will describe a strategy $\mathbf{st}$ for Generic in $\Game^{\rm rc\mathbf{2a}}_{\bar{\mu}}(p, \mathbb{P}^*)$. Whenever we say *Generic chooses x such that* we mean *Generic chooses the $<^*_\chi$–first x such that* (and likewise for other variants).

During a play of $\Game^{\rm rc\mathbf{2a}}_{\bar{\mu}}(p, \mathbb{P}^*)$ Generic constructs aside sequences $\langle p_\alpha : \alpha < \lambda\rangle$ and $\bar{\delta} = \langle \delta_\alpha : \alpha < \lambda\rangle$ so that for each $\alpha < \lambda$:

(a) $\bar{\delta}$ is a strictly increasing continuous sequence of ordinals below λ, $p_\alpha \in \mathbb{P}^*$ and $\{\delta_\xi : \xi \leq \omega + \alpha\} = C^{p_\alpha} \cap (\delta_{\omega+\alpha} + 1)$,
(b) if $\beta < \alpha$, then $p_\beta \leq p_\alpha$ and $\eta^{p_\alpha}{\restriction}\delta_{\omega+\beta} \vartriangleleft \eta^{p_\beta}$,
(c) $\{\delta_\xi : \xi \leq \omega\} = \{\delta \in C^p : {\rm otp}(\delta \cap C^p) \leq \omega\}$ and $p_0 = p$,
(d) $\delta_{\omega+\alpha+1}$ and $p_{\alpha+1}$ are determined right after stage α of $\Game^{\rm rc\mathbf{2a}}_{\bar{\mu}}(p, \mathbb{P}^*)$.

So suppose that the two players have arrived to a stage $\alpha < \lambda$ of a play of $\Game^{\rm rc\mathbf{2a}}_{\bar{\mu}}(p, \mathbb{P}^*)$, and Generic has constructed aside $\delta_{\omega+\beta+1}$ and $p_{\beta+1}$ for $\beta < \alpha$. If $\alpha = 0$ or it is a limit ordinal, then conditions (a)–(c) (and our rule of taking "the $<^*_\chi$–first") fully determine $\{\delta_\xi : \xi \leq \omega + \alpha\}$ and p_α.

Now Generic chooses an enumeration (without repetition) $\bar{\rho} = \langle \rho^\alpha_j : j < \mu_\alpha\rangle$ of ${\rm pos}(p_\alpha, \delta_{\omega+\alpha})$ such that $\rho^\alpha_0 = \eta^{p_\alpha}{\restriction}\delta_{\omega+\alpha}$. Antigeneric picks $\xi_\alpha < \lambda$ and the two players start a subgame of length $\mu_\alpha \cdot \xi_\alpha$. In the course of the subgame, in addition to her innings p^α_γ, Generic will also choose ordinals $\varepsilon^\alpha_\gamma = \varepsilon_\gamma < \lambda$ and sequences

$\varphi^\alpha_\gamma = \varphi_\gamma : \varepsilon_\gamma \longrightarrow \{-1,1\}$. These objects will satisfy the following demands (letting q^α_γ be the innings of Antigeneric):

(e) $\delta_{\omega+\alpha} \le \varepsilon_{\gamma'} \le \varepsilon_\gamma \in C^{q^\alpha_\gamma}$ and $\varphi_{\gamma'}{\restriction}[\delta_{\omega+\alpha},\varepsilon_{\gamma'}) = \varphi_\gamma{\restriction}[\delta_{\omega+\alpha},\varepsilon_{\gamma'})$ for $\gamma' < \gamma < \mu_\alpha \cdot \xi_\alpha$,

(f) if $\gamma = \mu_\alpha \cdot i + 2j$, $i < \xi_\alpha$ and $j < \mu_\alpha$, then
  - (i) $\rho^\alpha_j \trianglelefteq \varphi_\gamma \vartriangleleft \varphi_{\gamma+1}$, $\varphi_\gamma = \eta^{q^\alpha_\gamma}{\restriction}\varepsilon_\gamma$, and $\varphi_{\gamma+1}(\delta) = -\eta^{q^\alpha_{\gamma+1}}(\delta)$ for $\delta \in [\varepsilon_\gamma, \varepsilon_{\gamma+1})$,
  - (ii) $p^\alpha_0 = \rho_0 *_{\delta_{\omega+\alpha}} p_\alpha$ and $(\varphi_\gamma{\restriction}\varepsilon_{\gamma'}) *_{\varepsilon_{\gamma'}} q^\alpha_{\gamma'} \le p^\alpha_\gamma$ for $\gamma' < \gamma$, and
  - (iii) $q^\alpha_\gamma \le \varphi_\gamma *_{\varepsilon_\gamma} p^\alpha_{\gamma+1}$.

So suppose that the two players have arrived to a stage $\gamma = \mu_\alpha \cdot i + 2j$ $(i < \xi_\alpha$, $j < \mu_\alpha)$ of the subgame and $p^\alpha_{\gamma'}, q^\alpha_{\gamma'}, \varphi_{\gamma'}, \varepsilon_{\gamma'}$ have been determined for $\gamma' < \gamma$. Let $\varphi = \rho^\alpha_j{}^\frown \bigcup\limits_{\gamma'<\gamma} \varphi_{\gamma'}{\restriction}[\delta_{\omega+\alpha},\varepsilon_{\gamma'})$. It follows from (f) that the sequence $\langle(\varphi{\restriction}\varepsilon_{\gamma'} *_{\varepsilon_{\gamma'}} q^\alpha_{\gamma'} : \gamma' < \gamma\rangle$ is $\le_{\mathbb{P}^*}$–increasing, so Generic may choose an upper bound $p^\alpha_\gamma \in \mathbb{P}^*$ to it. (Note that necessarily $\varphi \vartriangleleft \eta^{p^\alpha_\gamma}$, $\sup(\varepsilon_{\gamma'} : \gamma' < \gamma) \le \min(C^{p^\alpha_\gamma})$.) She plays p^α_γ in the subgame and Antigeneric answers with $q^\alpha_\gamma \ge p^\alpha_\gamma$. Now Generic lets $\varepsilon_\gamma \in C^{q^\alpha_\gamma}$ be such that $|C^{q^\alpha_\gamma} \cap \varepsilon_\gamma| = 1$ and she puts $\varphi_\gamma = \eta^{q^\alpha_\gamma}{\restriction}\varepsilon_\gamma$ and she lets $\psi : \varepsilon_\gamma \longrightarrow \{-1,1\}$ be defined by $\psi{\restriction}\delta_{\omega+\alpha} = \rho_j$ and $\psi(\delta) = -\varphi_\gamma(\delta)$ for $\delta \in [\delta_{\omega+\alpha}, \varepsilon_\gamma)$. Then Generic plays $p^\alpha_{\gamma+1} = \psi *_{\varepsilon_\gamma} q^\alpha_\gamma$ as her inning at stage $\gamma+1$ of the subgame and Antigeneric answers with $q^\alpha_{\gamma+1} \ge p^\alpha_{\gamma+1}$. Finally, Generic picks $\varepsilon_{\gamma+1} \in C^{q^\alpha_{\gamma+1}}$ such that $|C^{q^\alpha_{\gamma+1}} \cap \varepsilon_{\gamma+1}| = 1$ and she takes $\varphi_{\gamma+1} : \varepsilon_{\gamma+1} \longrightarrow \{-1,1\}$ such that $\varphi_\gamma \vartriangleleft \varphi_{\gamma+1}$ and $\varphi_{\gamma+1}(\delta) = -\eta^{q^\alpha_{\gamma+1}}(\delta)$ for $\delta \in [\varepsilon_\gamma, \varepsilon_{\gamma+1})$. Plainly, the demands in (e)+(f) are satisfied (and both p^α_γ and $p^\alpha_{\gamma+1}$ are legal innings in $\partial^{\rm rc\mathbf{2a}}_{\bar{\mu}}(p,\mathbb{P}^*)$).

After the subgame is over, Generic lets $\varphi = \rho^\alpha_0{}^\frown \bigcup\{\varphi_\gamma{\restriction}[\delta_{\omega+\alpha},\varepsilon_\gamma) : \gamma < \mu_\alpha \cdot \xi_\alpha\}$, and she picks a $\le_{\mathbb{P}^*}$–upper bound $p'_{\alpha+1}$ to the increasing sequence $\langle(\varphi{\restriction}\varepsilon_\gamma) *_{\varepsilon_\gamma} q^\alpha_\gamma : \gamma < \mu_\alpha \cdot \xi_\alpha\rangle$. Note that $\varepsilon_\gamma \le \min\big(C^{p'_{\alpha+1}}\big)$ and $\varphi{\restriction}\varepsilon_\gamma \vartriangleleft \eta^{p'_{\alpha+1}}$ for all $\gamma < \mu_\alpha \cdot \xi_\alpha$, so also $\rho^\alpha_0 = \eta^{p_\alpha}{\restriction}\delta_{\omega+\alpha} \vartriangleleft \eta^{p'_{\alpha+1}}$. Also

(g) $(\eta^{p'_{\alpha+1}}{\restriction}\varepsilon_\gamma) *_{\varepsilon_\gamma} q^\alpha_\gamma \le p'_{\alpha+1}$ for all $\gamma < \mu_\alpha \cdot \xi_\alpha$.

Let $p_{\alpha+1} \in \mathbb{P}^*$ be such that $C^{p_{\alpha+1}} = \{\delta_\xi : \xi \le \omega+\alpha\} \cup C^{p'_{\alpha+1}}$ and $\eta^{p_{\alpha+1}} = \eta^{p'_{\alpha+1}}$ (plainly $p_\alpha \le p_{\alpha+1}$) and let $\delta_{\omega+\alpha+1} = \min\big(C^{p'_{\alpha+1}}\big)$.

This finishes the description of the strategy $\mathbf{st}$. Let us argue that $\mathbf{st}$ is a winning strategy for Generic. To this end suppose that

($\boxplus$) $\big\langle \xi_\alpha, \langle p^\alpha_\gamma, q^\alpha_\gamma : \gamma < \mu_\alpha \cdot \xi_\alpha\rangle : \alpha < \lambda\big\rangle$

is a result of a play of $\partial^{\rm rc\mathbf{2a}}_{\bar{\mu}}(p,\mathbb{P}^*)$ in which Generic follows $\mathbf{st}$ and the objects constructed aside are

$(\boxplus)^*_\alpha$ $p'_\alpha, p_\alpha, \delta_\xi, \langle\varepsilon^\alpha_\gamma, \varphi^\alpha_\gamma : \gamma < \mu_\alpha \cdot \xi_\alpha\rangle, \langle\rho^\alpha_j : j < \mu_\alpha\rangle$

(and the demands in (a)–(f) are satisfied). Let $C = \{\delta_\xi : \xi < \lambda\}$ (so it is a club of λ) and $\eta = \bigcup\limits_{\alpha<\lambda} \eta^{p_\alpha}{\restriction}\delta_\alpha$ (clearly $\eta : \lambda \longrightarrow \{-1,1\}$; remember (b)), and let $p^* = (\eta, C)$. It is a condition in $\mathbb{P}^*$ and it is stronger than all p_α (for $\alpha < \lambda$) so also $p^* \ge p$. Suppose that $\alpha < \lambda$ and $p' \ge p^*$. We will show that there is $p'' \ge p'$ such that for some $j < \mu_\alpha$, the condition p'' is stronger than all $q^\alpha_{\mu_\alpha \cdot i+j}$ for all $i < \xi_\alpha$. Without loss of generality, $\min(C^{p'}) \ge \delta_{\omega+\alpha+1}$. Let $j' < \mu_\alpha$ be such that $\eta^{p'}{\restriction}\delta_{\omega+\alpha} = \rho^\alpha_{j'}$. We consider two cases now.

CASE 1: $\eta^{p'}(\delta_{\omega+\alpha}) = \eta^{p^*}(\delta_{\omega+\alpha}) = \eta^{p_{\alpha+1}}(\delta_{\omega+\alpha})$.
Then $\eta^{p'}\restriction[\delta_{\omega+\alpha}, \delta_{\omega+\alpha+1}) = \eta^{p_{\alpha+1}}\restriction[\delta_{\omega+\alpha}, \delta_{\omega+\alpha+1})$. Let $j = 2\cdot j' < \mu_\alpha$, and we will argue that $q^\alpha_{\mu_\alpha\cdot i+j} \leq p'$ for all $i < \xi_\alpha$. So let $i < \xi_\alpha$, $\gamma = \mu_\alpha \cdot i + j$. By the choice of j' we know that $\eta^{p'}\restriction\delta_{\omega+\alpha} = \rho^\alpha_{j'} = \eta^{q^\alpha_\gamma}\restriction\delta_{\omega+\alpha}$ and also

$$\varphi^\alpha_\gamma\restriction[\delta_{\omega+\alpha}, \varepsilon^\alpha_\gamma) = \eta^{p'_{\alpha+1}}\restriction[\delta_{\omega+\alpha}, \varepsilon^\alpha_\gamma) = \eta^{p'}\restriction[\delta_{\omega+\alpha}, \varepsilon^\alpha_\gamma).$$

Hence (by (f)(i)) $\eta^{p'}\restriction\varepsilon^\alpha_\gamma = \eta^{q^\alpha_\gamma}\restriction\varepsilon^\alpha_\gamma$ and now

$$\begin{aligned}&q^\alpha_\gamma \leq (\eta^{p'}\restriction\varepsilon^\alpha_\gamma) *_{\varepsilon_\gamma} q^\alpha_\gamma \leq (\eta^{p'}\restriction\delta_{\omega+\alpha}) *_{\delta_{\omega+\alpha}} p'_{\alpha+1} = (\eta^{p'}\restriction\delta_{\omega+\alpha+1}) *_{\delta_{\omega+\alpha+1}} p'_{\alpha+1} = \\ &(\eta^{p'}\restriction\delta_{\omega+\alpha+1}) *_{\delta_{\omega+\alpha+1}} p_{\alpha+1} \leq (\eta^{p'}\restriction\delta_{\omega+\alpha+1}) *_{\delta_{\omega+\alpha+1}} p^* \leq (\eta^{p'}\restriction\delta_{\omega+\alpha+1}) *_{\delta_{\omega+\alpha+1}} p' = p'\end{aligned}$$

(for the second inequality remember (g)).
CASE 2: $\eta^{p'}(\delta_{\omega+\alpha}) = -\eta^{p^*}(\delta_{\omega+\alpha}) = -\eta^{p_{\alpha+1}}(\delta_{\omega+\alpha})$.
Then $\eta^{p'}(\delta) = -\eta^{p^*}(\delta) = -\eta^{p_{\alpha+1}}(\delta)$ for all $\delta \in [\delta_{\omega+\alpha}, \delta_{\omega+\alpha+1})$. Let $j = 2\cdot j' + 1$ and let us argue that $q^\alpha_{\mu_\alpha\cdot i+j} \leq p'$ for all $i < \xi_\alpha$. So let $i < \xi_\alpha$, $\gamma = \mu_\alpha \cdot i + j$. Like in the previous case we show that $\eta^{p'}\restriction\varepsilon^\alpha_\gamma = \eta^{q^\alpha_\gamma}\restriction\varepsilon^\alpha_\gamma$ and then easily

$$q^\alpha_\gamma \leq (\eta^{p'}\restriction\varepsilon_\gamma) *_{\varepsilon^\alpha_\gamma} q^\alpha_\gamma \leq (\eta^{p'}\restriction\delta_{\omega+\alpha+1}) *_{\delta_{\omega+\alpha+1}} p'_{\alpha+1} \leq (\eta^{p'}\restriction\delta_{\omega+\alpha+1}) *_{\delta_{\omega+\alpha+1}} p' = p'.$$

□

□

Proposition 4.4. *Let $\underset{\sim}{\eta}$ be a $\mathbb{P}^*$–name such that $\Vdash_{\mathbb{P}^*} \underset{\sim}{\eta} = \bigcup\{\eta^p\restriction\min(C^p) : p \in \Gamma_{\mathbb{P}^*}\}$. Then $\Vdash_{\mathbb{P}^*}$ " $\underset{\sim}{\eta} :\longrightarrow \{-1,1\}$ " and for every $s \in \mathbb{Q}^0_\lambda \cap \mathbf{V}$,*

$$\Vdash_{\mathbb{P}^*} \text{“ } \{\alpha < \lambda : \underset{\sim}{\eta}(\alpha) = -1\} \in \mathrm{fil}(s)^+ \ \textit{ and } \ \{\alpha < \lambda : \underset{\sim}{\eta}(\alpha) = 1\} \in \mathrm{fil}(s)^+ \text{ ”}.$$

Proof. It should be clear that $\Vdash_{\mathbb{P}^*}$ " $\underset{\sim}{\eta} :\longrightarrow \{-1,1\}$ ", so let us show the second statement. Assume $p \in \mathbb{P}^*$, $s \in \mathbb{Q}^0_\lambda$. Choose a continuous increasing sequence $\langle\delta_\xi : \xi < \lambda\rangle \subseteq C^p$ such that for every $\xi < \lambda$ there is $\alpha = \alpha(\xi) \in C^s$ such that $Z^s_\alpha \subseteq [\delta_\xi, \delta_{\xi+1})$. Then let $C = \{\delta_\xi : \xi < \lambda \text{ is even }\}$ (it is a club of λ) and let $\eta : \lambda \longrightarrow \{-1,1\}$ be such that

- $\eta\restriction[\delta_\xi, \delta_{\xi+1}) \in \big\{\eta^p\restriction[\delta_\xi, \delta_{\xi+1}), -\eta^p\restriction[\delta_\xi, \delta_{\xi+1})\big\}$,
- if $\xi < \lambda$ is even, then $\big\{\delta \in Z^s_{\alpha(\xi)} : \eta(\delta) = 1\big\} \in d^s_{\alpha(\xi)}$,
- if $\xi < \lambda$ is odd, then $\big\{\delta \in Z^s_{\alpha(\xi)} : \eta(\delta) = -1\big\} \in d^s_{\alpha(\xi)}$.

Now note that $(\eta, C) \in \mathbb{P}^*$ is a condition stronger than p and

$$(\eta, C) \Vdash_{\mathbb{P}^*} \text{“ } \big\{\alpha < \lambda : \underset{\sim}{\eta}(\alpha) = 1\big\} \in \mathrm{fil}(s)^+ \text{ and } \big\{\alpha < \lambda : \underset{\sim}{\eta}(\alpha) = -1\big\} \in \mathrm{fil}(s)^+ \text{ ”}.$$

□

Corollary 4.5. *Assume λ is a strongly inaccessible cardinal. Then there is a forcing notion $\mathbb{P}$ such that*

$\Vdash_{\mathbb{P}}$ " *λ is strongly inaccessible and $2^\lambda = \lambda^{++}$ and*
there is no very reasonable ultrafilter on λ with a generating system of size $< 2^\lambda$ "

Proof. We may start with the universe $\mathbf{V}$ in which $2^\lambda = \lambda^+$. Let $\bar{\mathbb{Q}} = \langle\mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \lambda^{++}\rangle$ be a λ–support iteration of the forcing notion $\mathbb{P}^*$ (see Definition 4.1). This forcing is nicely double **a**–bounding over $\bar{\mu}$ (where $\mu_\alpha = 2^{|\alpha|+\aleph_0}$; remember Proposition 4.3) and hence $\mathbb{P}_{\lambda^{++}}$ is nicely double **a**–bounding over $\bar{\mu}$ (by Theorem

2.12). Using Theorem 2.2 we conclude that $\mathbb{P}_{\lambda^{++}}$ does not collapse any cardinals and forces that $2^\lambda = \lambda^{++}$. Proposition 4.4 implies that

$$\Vdash_{\mathbb{P}_{\lambda^{++}}} \text{" for no family } G^* \subseteq \mathbb{Q}^0_\lambda \text{ of size } < 2^\lambda, \text{fil}(G^*) \text{ is an ultrafilter on } \lambda \text{ ".}$$

□

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